MATH 6: HANDOUT 19
ARITHMETIC SEQUENCES

Arithmetic Sequences

A sequence of numbers is an **arithmetic sequence** or **arithmetic progression** if the difference between consecutive terms is the same number, the **common difference** or d.

Example: The sequence $1, 5, 9, 13, 17, \ldots$ is an arithmetic sequence because the difference between consecutive terms is $d = 4$.

We can also find the n-th term if we know the 1st term and d?

Example: What is a_{100} in the example above?

$$\begin{aligned}
&a_1 = 1\\
&a_2 = a_1 + d = 1 + 4 = 5\\
&a_3 = a_2 + d = (a_1 + d) + d = a_1 + 2d = (1 + 4) + 4 = 1 + 2 \times 4 = 9\\
&a_4 = a_3 + d = (a_2 + d) + d = ((a_1 + d) + d) + d = a_1 + 3d = 1 + 3 \times 4 = 13
\end{aligned}$$

The pattern is:

$$\begin{aligned}
a_n &= a_1 + (n-1)_d\\
a_{100} &= a_1 + 99d = 1 + 99 \times 4 = 397
\end{aligned}$$

Properties of an Arithmetic Sequence

A useful property of an arithmetic sequence is that any term is the arithmetic mean of its neighbors:

$$a_n = \frac{a_{n-1} + a_{n+1}}{2}$$

Proof:

$$\begin{aligned}
a_n &= a_{n-1} + d\\
a_n &= a_{n+1} - d
\end{aligned}$$

Adding these two equalities gives us:

$$2a_n = a_{n-1} + a_{n+1}$$

from where we can get what we need.

Another property of arithmetic sequences is that we can find the common difference d if we know any two terms a_m and a_n:

$$d = \frac{a_m - a_n}{m - n}$$

Sum of an Arithmetic Sequence

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n = n \times \frac{a_1 + a_n}{2}$$

Proof: To prove this, we write the sum in 2 ways, in increasing and decreasing order:

$$\begin{aligned}
S_n &= a_1 + a_2 + \cdots + a_n\\
S_n &= a_n + a_{n-1} + \cdots + a_1
\end{aligned}$$

Adding these two expressions up and noticing that $a_1 + a_n = a_2 + a_{n-1} = a_3 + a_{n-2} = \ldots$ we get:

$$\begin{aligned}
2S_n &= (a_1 + a_n) \times n\\
S_n &= n \times \frac{a_1 + a_n}{2}
\end{aligned}$$