Ancient Mathematics: Pythagoras

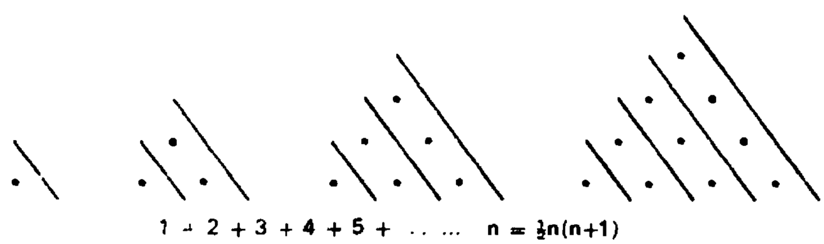
"The so-called Pythagoreans applied themselves to mathematics, and were the first to develop this science; and through studying it they came to believe that its principles are the principles of everything." – Aristotle's Metaphysics, Book 1, 985b

Mathematics in Ancient Greece was set apart from its predecessors in Egypt and Babylon by emphasis on generalization, and its foundation on axioms and rigorous proof. The age of the Pythagoreans is thus of particular importance to the history and development of modern mathematics, see [1].

Pythagoras himself, although thought of as a mathematician today, was at the time regarded in a different light. Herodotus called him an "important sophist", a paid teacher of philosophy, rhetoric and science in ancient Greece. Others thought of him as the founder of a religious order based on the principle that "*All is number.*"[1]

The Pythagoreans studied laws of *whole* numbers. Indeed, in common with other ancient cultures, the only numbers in Greek mathematics were positive integers. The Pythagoreans pictured integers as constellations – groups of points arranged in some geometrical pattern. They wished to express geometrical shapes as numbers, in service of their belief that all matter could be formed from basic shapes.

We study first triangular numbers



Triangular numbers

Notice the pattern: the rows of the triangle contain 1, 2, 3, 4, etc points. The next row always increases by one point. Thus, the nth triangle (where n

[1]They were vegetarians, believed in the transmigration of souls, and accepted women as students. As part of this religion, apparently, Pythagoras prohibited the consumption of fava beans.

is a whole number) has

$$T_n := 1 + 2 + 3 + \cdots + n$$

points. Recall how to find this sum:

$$\begin{aligned} T_n + T_n = 1 + &\quad 2 \quad + \quad 3 \quad + \cdots + n \\ &+ n + (n-1) + (n-2) + \cdots + 1 = n(n+1). \end{aligned}$$

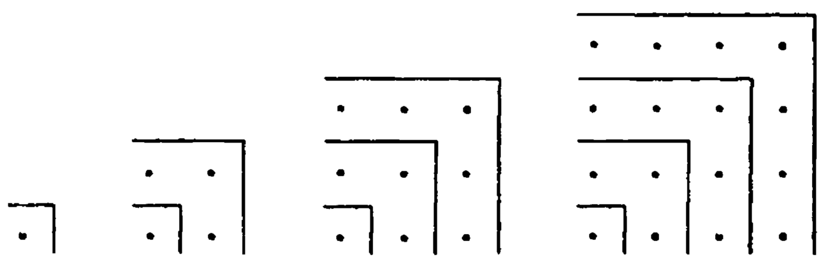
Thus $T_n = \frac{1}{2}n(n+1)$ gives a formula for the nth triangular number. The first few are $T_1 = 1$, $T_2 = 3$, $T_3 = 6$, $T_4 = 10$, $T_5 = 15$, $T_6 = 21$, $T_7 = 28$, etc.

Even more remarkable are the numbers from a square array. Here, if the nth square has n points on its side, the next square has $n+1$ points which means an addition of $n + n + 1 = 2n + 1$ points total. So

$$S_n = 1 + 3 + 5 + \cdots + (2n-1) = n^2.$$

Since to get S_{n+1}, we add $2n+1$ points, we obtain the the remarkable conclusion: differences between successive squares give the sequence of odd numbers:

$$(n+1)^2 - n^2 = 2n + 1. \tag{1}$$



$1 + 3 + 5 + 7 + \cdots (2n-1) = n^2$

Square numbers

With the aid of the formula (1) , we can discover sets of numbers that satisfy the **Pythagorean equation**:

$$a^2 + b^2 = c^2. \tag{2}$$

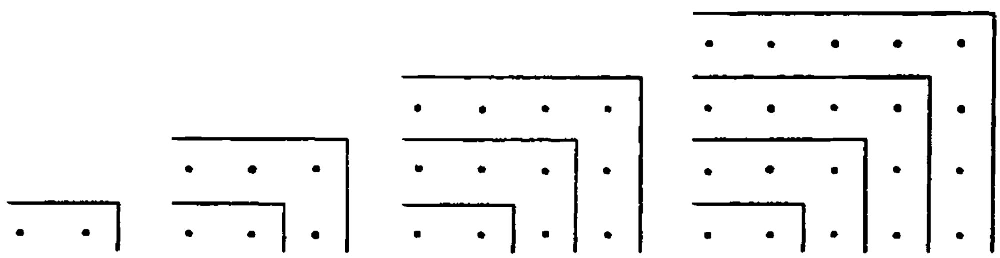
To do so, we must make $2n+1$ into a square, say m^2. In this case, $n = \frac{1}{2}(m^2 - 1)$ and $n + 1 = \frac{1}{2}(m^2 + 1)$.[2] Thus (1) becomes

$$m^2 + \left(\frac{m^2-1}{2}\right)^2 = \left(\frac{m^2+1}{2}\right)^2.$$

m	a	b	c
3	3	4	5
5	5	12	13
7	7	24	25
9	9	40	41

Now, m must be odd since m^2 is odd. Thus, plugging in $m = 3, 5, 7, 9$ we find

So far we have discovered that successive positive integers leads to triangular numbers, and successive odd integers leads to squares. *What about if we add successive even integers?*. This can be seen by considering a rectangular configuration of sides n and $n+1$ dots. This next rectangle has $n + n = 2n$



Rectangular numbers

more dots than the previous.

$$R_n = 2 + 4 + 6 + 8 + \cdots + 2n = 2(1 + 2 + 3 + 4 + \cdots + n) = n(n+1) = 2T_n.$$

Thus, the rectangle can be divided into two triangles, along the "diagonal" (not so with the square). So, adding odd numbers leads to a square array whereas adding even numbers leads to a rectangular array in which the ratio $(n+1)/n$ of the sides depends on n (unlike the square). Thus, the Pythagoreans deduced the following correspondence:

$$\text{odd} \iff \text{limited} \qquad\qquad \text{even} \iff \text{unlimited}$$

Today, we would not draw such conclusions.

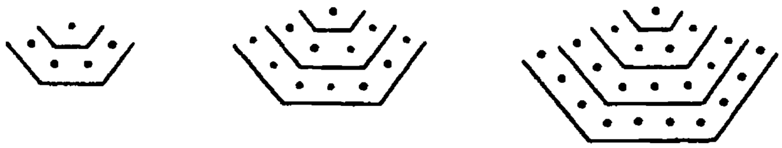
Finally, we study pentagonal numbers. Here, there are $n+n+n-2 = 3n-2$ additional points added each step

$$P_n = 1 + 4 + 7 + 10 + \cdots + (1 + 3(n-1)) = \frac{1}{2}n(3n-1).$$

[2]Note that m^2 is odd, so the fractions defining n and $n+1$ are integers

Note that the nth pentagonal number is one third of the $(3n-1)$th triangular number. Moreover, there are the interesting identities:

$$P_n = T_{n-1} + S_n = T_{2n-1} - T_n.$$



$1 + 4 + 7 + 10 + \ldots\ldots (3n-2) = \frac{1}{2}n(3n-1)$

Pentagonal numbers

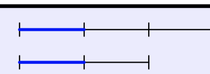
The discoveryof the relation between musical harmony and the theory of proportion, attributed to Pythagoras; reinforced this view of the structure of the universe:

- Identical strings whose lengths are in the ratio 2:1 vibrate an octave apart.
- A perfect fifth corresponds to the ratio 3:2.
- A perfect fourth corresponds to the ratio 4:3.

The physical line segments and triangles of Pythagorean geometry could similarly be considered to be made up of discrete numerical elements.

Let us now discuss this issue of commensurability and incommensurability The Pythagorean commensurability supposition stems from their basic tenets:

Definition. Lengths are in the ratio $m:n$ if some sub-length divides exactly m times into the first and n times into the second.
Lengths are *commensurable* if some sub-length divides exactly into both.

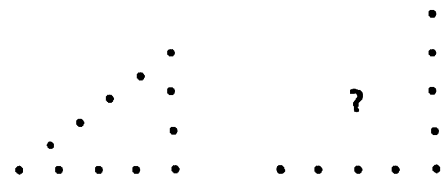


Ratio 3 : 2

all is number and that the design of the gods be perfect (whole).

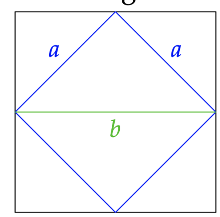
This picture was destroyed when it was discovered that the diagonal of the square was incommensurable with the side. In this case, if the side be constructed of a finite number of discrete elements, how can the hypotenuse be constructed?

The discovery of incommensurable ratios produced a crisis; apparently a disciple named Hippasus (c. 500 BC) was set adrift at sea as punishment for its revelation. However, by 340 BC, the Greeks were happy to admit the existence of incommensurable lengths.



Theorem (Aristotle): If the diagonal and the side of the square are commensurable, then odd numbers equal even numbers.

Inferred proof. In Socrates' doubled-square, suppose that side : diagonal $= a : b$; these are integers! Assume at least one of a or b is odd, else the common sub-length may be doubled. The larger square is twice the smaller, whence the square numbers have ratio



$$b^2 : a^2 = 2 : 1$$

It follows that b^2 is even and thus divisible by 4, whence a^2 is also even, and both a, b are even. Whichever of a, b was odd is also even: contradiction! ∎

Note the similarity of this argument to the modern proof of the irrationality of $\sqrt{2}$.

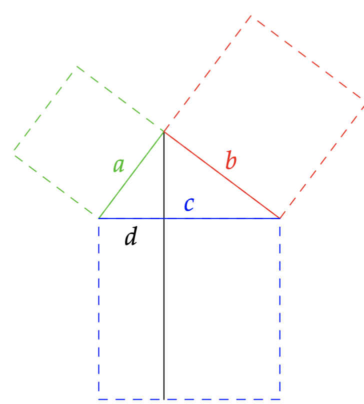
Indeed, we have the famous Pythagorean theorem

Theorem (Pythagoreans): If a, b and c represent the side lengths of a right triangle, c being the hypotenuse, then $a^2 + b^2 = c^2$.

'Proof' of Pythagoras' Theorem. Label the right triangle a, b, c where c is the hypotenuse and drop the altitude to the hypotenuse. Let d be the length shown. Similar triangles tell us that

$$a : d = c : a \implies a^2 : ad = cd : ad \implies a^2 = cd$$

Thus the square on a has the same area as the rectangle below the d-side of the hypotenuse. Repeat the calculation on the other side to obtain $b^2 = c(c - d)$, and sum to complete the proof. ∎



Supposing $a = b = 1$, follows that $\sqrt{2}$ is an irrational number.

Note: this argument very possiblty that of the Pythagoreans. However, since the only numbers were integers, the symbols a, b, c, d are integer multiples of an assumed common sub-length for the Pythagoreans. This restriction destroys the generality of the argument. Book I of Euclid's Elements had the primary goal to provide a rigorous proof of Pythagoras' Theorem which did not depend on commensurability. Note that, with our modern understanding of real numbers, there is nothing wrong with the above argument.

References

[1] Meschkowski, Herbert. "Ways of thought of great mathematicians: an approach to the history of mathematics." (1964).

[2] Baron, Margaret E. The origins of infinitesimal calculus. Elsevier, 2014.