MATH 5: HANDOUT

The Product Rule

When two events are **independent** (one doesn't affect the other), multiply their probabilities:

Theorem 1 (Product Rule)**.** For independent events:

$$P(A \text{ and } B) = P(A) \times P(B)$$

For n independent events: $P(A_1 \text{ and } A_2 \text{ and } \cdots \text{ and } A_n) = P(A_1) \times P(A_2) \times \cdots \times P(A_n)$

Example: Roll two dice. $P(5 \text{ and } 6)$?

$$P = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$$

Example: Toss a coin 3 times. $P(\text{HHH})$?

$$P = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

What Does "Independent" Mean?

Events are **independent** if knowing one outcome gives no information about the other.

- **Independent:** Rolling dice, flipping coins, spinning spinners—each trial "starts fresh."
- **Not independent:** Drawing cards *without replacement*—fewer cards remain after each draw.

Probabilities Shrink Fast!

Coin flips	P(all heads)	Approx.
5	$\frac{1}{32}$	1 in 32
10	$\frac{1}{1024}$	1 in 1,000
20	$\frac{1}{1048576}$	1 in a million

"At Least One" Problems

Finding $P(\text{at least one success})$ directly is hard. Use the complement instead:

Theorem 2 (At Least One Strategy)**.** $P(\text{at least one success}) = 1 - P(\text{no successes})$

Example: Toss a coin 10 times. $P(\text{at least one heads})$?

$$P(\text{at least one H}) = 1 - P(\text{all tails}) = 1 - \left(\frac{1}{2}\right)^{10} = 1 - \frac{1}{1024} = \frac{1023}{1024} \approx 99.9\%$$

Counting with Repetition Allowed

When making a sequence of choices, multiply the number of options at each step.

Theorem 1 (Multiplication Principle)**.** If you choose k items from n options, with repetition allowed:

$$\text{Number of ways} = n^k$$

Example: 3-letter words from 26 letters?

$$26^3 = 17{,}576$$

Example: 4-digit PIN codes (0–9)?

$$10^4 = 10{,}000$$

Factorials

Definition 1 (Factorial)**.** $n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$ (by convention, $0! = 1$)

Factorials grow fast: $5! = 120$, $10! \approx 3.6$ million, $20! \approx 2.4$ quintillion
To arrange all n items in order: $n!$ ways.

Counting Without Repetition (Permutations)

When each item can only be used once, the options decrease at each step.

Definition 2 (Permutations)**.** Choosing k items from n options, **without repetition**, order matters:

$$P(n,k) = n \times (n-1) \times (n-2) \times \cdots \times (n-k+1) = \frac{n!}{(n-k)!}$$

Example: Gold, silver, bronze medals for 10 runners? $P(10,3) = 10 \times 9 \times 8 = 720$

Summary: With vs. Without Repetition

	With Repetition	**Without Repetition**
Rule	Can reuse items	Each item used at most once
Formula	n^k	$P(n,k) = n(n-1)(n-2)\cdots$
Example	PIN codes, passwords	Medals, officer positions

Homework

1. Calculate:
   (a) $6!$
   (b) $\frac{9!}{7!}$
   (c) $P(8,4)$
   (d) $P(10,2)$

2. At a fair, you toss small balls into a crate full of bottles. Each ball has a 20% probability of landing inside a bottle. You win if at least one ball lands inside.
(a) If you get 3 balls, what is the probability of winning?
(b) If you get 5 balls, what is the probability of winning?
(c) They charge $2 for 3 balls or $3 for 5 balls. Which is the better deal (considering only the probability of winning)?

3. A group of 6 friends always dine at the same table with exactly 6 chairs. They decide to sit in a different arrangement each day.
   (a) How many different arrangements are possible?
   (b) Can they keep this up for a whole year without repeating?

4. In roulette, there are 37 slots (0 through 36). Among 1–36, half are red and half are black (zero has no color). Find the probability of:
(a) Getting red on a single spin
(b) Getting red, then black, then 0 on three consecutive spins
(c) Getting red 5 times in a row

5. How many ways are there to seat 15 students in a classroom with:
   (a) Exactly 15 chairs?
   (b) 25 chairs?