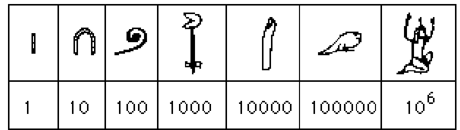
Math 5b. Classwork 4.

Numeral systems.

Over the long centuries of human history, many different numeral systems have appeared in different cultures. The oldest systems weren't a place-valued system.

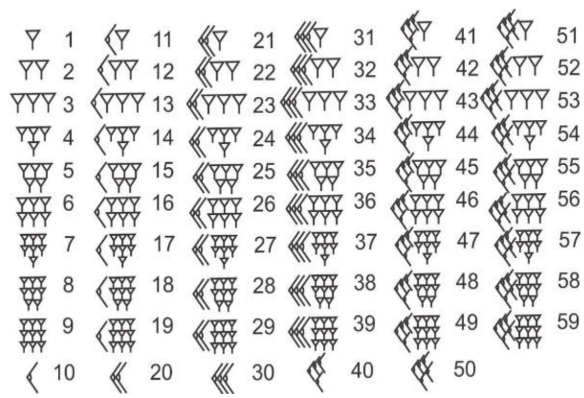
The good example of such system is the ancient Egyptian decimal numeral system.



Egyptian numeral hieroglyphs

It's decimal, but numbers can be written in an arbitrary order, but usually were red from right to left and/or from bottom to top.

1,333,330 in Egyptian hieroglyphs from the Edfu Temple (237–57 BCE) in Egypt.



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The Babylonian system (form about 2000 BC) used only two symbols to write any number between 1 and 60, which shows that it was based on 60 and had some 10 biased system inside:

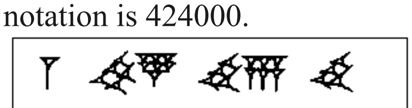
Also, it was a position value system, for example, number 62 was shown as  , which means one time 60 and 2. The use of this sexagesimal (60-based) system is still noticeable now, we have 60 minutes in one hour, 60 seconds in a minute, 360° for the full turn around.

Now we adopt a notation where we separate the numerals by commas when we use our digits to represent Babylonian notation, for example, 1,57,46,40 represents the sexagesimal number $60^3 \cdot 1 + 60^2 \cdot 57 + 60^1 \cdot 46 + 60^0 \cdot 40$which, in decimal notation is 424000.

Babylonian mathematical tablet Plimpton 322.Credit...Christine Proust and Columbia University

Here is **1,57,46,40 in Babylonian numerals**

https://mathshistory.st-andrews.ac.uk/HistTopics/Babylonian_mathematics/



1,57,46,40 = 424000

Another very well-known numeral system is roman; it was used for thousands of years and in some cases is still used now as well. It's a “decimal”, 10 based system, but the symbols (letters) are used in an unusual way.

Symbol	I	V	X	L	C	D	M
Value	1	5	10	50	100	500	1000

1	2	3	4	5	6	7	8	9	10
I	II	III	IV	V	VI	VII	VIII	IX	X

For example, 4 is one less than 5, so 4 can be written as IV. Same principle of subtractive notation is used for 9 -> IX, 40 and 90 -> XL and XC, 400 and 900 -> CD and CM
Some other examples:

- 29 = XX + IX = **XXIX**.
- 347 = CCC + XL + VII = **CCCXLVII**.
- 789 = DCC + LXXX + IX = **DCCLXXXIX**.
- 2,421 = MM + CD + XX + I = **MMCDXXI**

Any missing place (represented by a zero in the place-value equivalent) is omitted, as in Latin (and English) speech:

- 160 = C + LX = **CLX**
- 207 = CC + VII = **CCVII**
- 1,009 = M + IX = **MIX**
- 1,066 = M + LX + VI = **MLXVI**

Can non-decimal place-value system be created? For example, with base 5?
Let see, how we can create this kind of system (we use our normal digits).

Num_{10}	1	2	3	4	5	6	7	8	9	10
Num_5	1	2	3	4	10	11	12	13	14	20

11	12	13	14	15	16	17	18	19	20
21	22	23	24	30	31	32	33	34	40

21	22	23	24	25	26	27	28	29	30
41	42	43	44	100	101	102	103	104	105

We only have 5 digits (0, 1, 2, 3, 4), and 4 first "natural" numbers in such system will be represented as one-digit numbers. Number 5 then should be shown as a 2-digit number, with fist digit 1 (place - value equal to 5^1) and 0 of "units". Any number is now written in the form

$$\dots + 5^3 \cdot n + 5^2 \cdot m + 5^1 \cdot k + 5^0 \cdot p, \qquad n, m, k, p \quad \text{are} \quad 0, 1, 2, 3, 4$$

$33 = 25 + 5 + 3 = 5^2 \cdot 1 + 5^1 \cdot 1 + 5^0 \cdot 3 = 113_5$

$195 = 125 + 25 \cdot 2 + 5 \cdot 4 = 5^3 \cdot 1 + 5^2 \cdot 2 + 5^1 \cdot 4 + 0 = 1240_5$

And vice versa, we can transform the number form 5-base to decimal system:

$2312_5 = 5^3 \cdot 2 + 5^2 \cdot 3 + 5^1 \cdot 1 + 2 = 250 + 75 + 5 + 2 = 332$

Let's do the addition of 1240_5 and 2312_5 (I omitted $_5$ notation):

$$\begin{array}{r} 1240 \\ +\underline{2312} \\ 4102 \end{array}$$

$4102_5 = 5^3 \cdot 4 + 5^2 \cdot 1 + 5^1 \cdot 0 + 2 = 125 \cdot 4 + 25 \cdot 1 + 0 + 2 = 500 + 25 + 2 = 527$

Let's try to introduce a new digit S for 10 and then create an 11 based system.

$11^2 = 121, \qquad 11^3 = 1331$

$890 = 121 \cdot 7 + 11 \cdot 3 + 10 = 11^2 \cdot 7 + 11^1 \cdot 3 + 10 = 73S_{11}$

$4S2_{11} = 11^2 \cdot 4 + 11^1 \cdot 10 + 2 = 484 + 110 + 2 = 596$

There is another very important place-value system: binary system, base 2 system where only two digits exist; 0, and 1.

Num_{10}	1	2	3	4	5	6	7	8	9	10
Num_2	1	10	11	100	101	110	111	1000	1001	1010

In this system place value of a digit is a power of 2:

$$\dots + 2^3 \cdot (0,1) + 2^2 \cdot (0,1) + 2^1 \cdot (0,1) + 2^0 \cdot (0,1)$$

$11 = 8 + 2 + 1 = 2^3 \cdot 1 + 2^2 \cdot 0 + 2^1 \cdot 1 + 2^0 \cdot 1 = 1011_2$

$75 = 64 + 8 + 2 + 1 = 2^6 \cdot 1 + 2^5 \cdot 0 + 2^4 \cdot 0 + 2^3 \cdot 1 + 2^2 \cdot 0 + 2^1 \cdot 1 + 2^0 \cdot 1 = 1001011_2$

$$\begin{aligned} 189 &= 128 + 32 + 16 + 8 + 4 + 1 \\ &= 2^7 \cdot 1 + 2^6 \cdot 0 + 2^5 \cdot 1 + 2^4 \cdot 1 + 2^3 \cdot 1 + 2^2 \cdot 1 + 2^1 \cdot 0 + 2^0 \cdot 1 \\ &= 10111101_2 \end{aligned}$$

Binary numbers can be transferred to decimals too. We will do it from right to left for convenience.

$$11010111_2 = 2^0 \cdot 1 + 2^1 \cdot 1 + 2^2 \cdot 1 + 2^3 \cdot 0 + 2^4 \cdot 1 + 2^5 \cdot 0 + 2^6 \cdot 1 + 2^7 \cdot 1$$
$$= 128 + 64 + 16 + 4 + 2 + 1 = 215$$

Exercises:

1. Write the numbers, written in the binary system in decimal system:

$$a. 11011011_2; \qquad b. 10001101_2, \qquad c. 111111111_2$$

2. Write the numbers 238 and 195 in binary system.

3. Write the numbers 245 and 324 in 6-based place-value system. Remember, that in this system you will have only 0, 1, 2, 3. 4, and 5 as digits.

4. Write numbers 234 in ternary system.

5. Write the number 212210_3 in decimal system.

6. Write the numbers 234_6 and 403_6 written in the 6-based place-value system (small number 6 shows that the number is not in decimal, but in 6-based system) in decimal system.

7. Write 673 in base-6 system.

8. Robert thought of a number not less than 1 and not more than 1000. Julia is allowed to ask only such questions to which Robert can answer “yes” or “no” (Robert always tells the truth). Can Julia determine the hidden number in 10 questions?

9. There is a bag of sugar, a scale and a weight of 1 g. Is it possible to measure 1 kg of sugar in 10 weights?