Math 5b. Classwork 3.

If a number $a$ in a power $n$ is divided by the same number in a power $m$,

$$\frac{a^n}{a^m} = \frac{\overbrace{a \cdot a \cdot \ldots \cdot a}^{n\ times}}{\underbrace{a \cdot a \cdot \ldots \cdot a}_{m\ times}} = \left(\underbrace{a \cdot a \cdot \ldots \cdot a}_{n\ times}\right) : \left(\underbrace{a \cdot a \cdot \ldots \cdot a}_{m\ times}\right) = a^n : a^m = a^{n-m}$$

So, we can say that

$$a^{-1} = \frac{1}{a^1} = \frac{1}{a}; \qquad a^{-n} = \frac{1}{a^n};$$

Let's see how our decimal system of writing numbers works when we use the concept of exponent:

$$3456 = 1000 \cdot 3 + 100 \cdot 4 + 10 \cdot 5 + 1 \cdot 6 = 10^3 \cdot 3 + 10^2 \cdot 4 + 10^1 \cdot 5 + 10^0 \cdot 6$$

The value of a place of a digit is defined by a power of 10 multiplied by the digit. Very large numbers can be written using this system, as well as very small numbers.

$$0.3 = \frac{1}{10} \cdot 3 = \frac{1}{10^1} = 10^{-1} \cdot 3$$

$$24.345 = 10^1 \cdot 2 + 10^0 \cdot 4 + 10^{-1} \cdot 3 + 10^{-2} \cdot 4 + 10^{-3} \cdot 5$$

Scientists work with very large and very small things, from galaxies to viruses. They need to be able to write numbers, describing the object of interest, for example the distance between two galaxies or the diameter of a virus.

One of the most important numbers in the universe is the speed of light.

299 792 458 m / s. It's very convenient to represent it as a decimal starting with units and multiplied by a power of 10.

$$299\ 792\ 458\ m\ per\ s = 2.99792458 \cdot 10^8\ m\ p\ s.$$

Let's convert the value to kilometers per hour. Each kilometer is 1000 meters, so we need to divide it by 1000:

$$3 \cdot \frac{10^8}{10^3} mps = 3 \cdot 10^{8-5} = 3 \cdot 10^5 km\ per\ s$$

In each hour there are 3600 seconds, or $3.6 \cdot 10^3$ seconds. To find out the speed of light in km per hour we now need to multiply the speed in seconds by $3.6 \cdot 10^3$

$$3 \cdot \frac{10^8}{10^3} mps = 3 \cdot 10^{8-5} = 3 \cdot 10^5 km\ per\ s = 3 \cdot 10^5 \cdot 3.6 \cdot 10^3 = 10.8 \cdot 10^8\ km\ p\ h$$

The Milky Way galaxy has a diameter of 105,700 light years, so the light will travel from one end to the other through its center in 105700 years.

How far is one side from the other in the Milky Way in kilometers? 10.8·108$km\ p\ h$ ·105700 $years$.

How many hours in a year? $24 \cdot 365.25 = 8766 \approx 8.8 \cdot 10^3\ hours$

$$10.8 \cdot 10^8\ km\ p\ h\ \cdot 105700\ years \approx 1.08 \cdot 10^9\ km\ p\ h \cdot 8.8 \cdot 10^3 \cdot 1.06 \cdot 10^5$$
$$\approx 10.07 \cdot 10^{17} km \approx 10^{18} km.$$

This way to write numbers is called scientific notation, it's used a lot in science for describing various objects, big and small. The number should be represented as a product of one-digit whole part of a number with decimal part rounded up to two digit after the point and a power of 10. For example, the distance from the Earh to the Moon is about $238{,}855$ miles (384,400 kilometers).

$$238855 = 2.38855 \cdot 10^5 \approx 2.39 \cdot 10^5\ miles$$

There's a famous legend about the origin of chess that goes like this. When the inventor of the game showed it to the emperor of India, the emperor was so impressed by the new game, that he said to the man

"*Name your reward!*"

The man responded,

"*Oh emperor, my wishes are simple. I only wish for this. Give me one grain of rice for the first square of the chessboard, two grains for the next square, four for the next, eight for the next and so on for all 64 squares, with each square having double the number of grains as the square before.*"

The emperor agreed, amazed that the man had asked for such a small reward - or so he thought. After a week, his treasurer came back and informed him that the reward would add up to an astronomical sum, far greater than all the rice that could conceivably be produced in many centuries!

$$2 + 2^2 + 2^3 + 2^4 + \cdots + 2^{64} = 36893488147419103230$$

This number is huge! We can write it as $3.6893488147419103230 \cdot 10^{19} \approx 3.69 \cdot 10^{19}$

Let's take a look on the small things, like bacteria and viruses.

$$1cm = 0.01m = \frac{1}{100}m = \frac{1}{10^2}m = 10^{-2}m$$

$$1mm = 0.001m = \frac{1}{1000}m = \frac{1}{10^3}m = 10^{-3}m$$

$$1\mu m = 10^{-6}m, \quad 1nm = 10^{-9}m$$

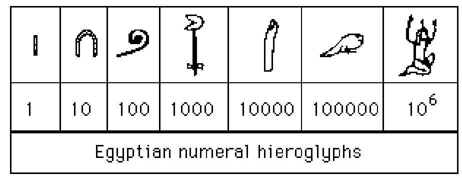
Bactria are between $0.5 - 1.5\ \mu m$

$$0.5\mu m = 0.5 \cdot 10^{-6}m = 5 \cdot 10^{7}m$$

**Numeral systems.**

Over the long centuries of human history, many different numeral systems have appeared in different cultures. The oldest systems weren't a place-valued system.

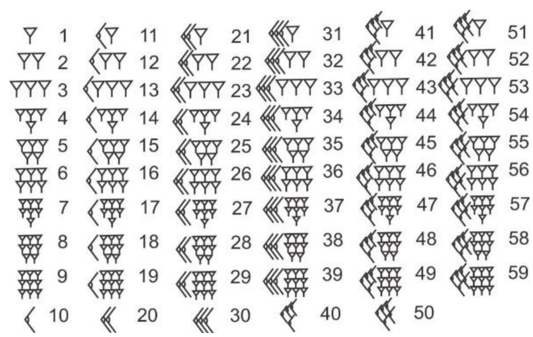
The good example of such system is the ancient Egyptian decimal numeral system.




It's decimal, but numbers can be written in an arbitrary order, but usually were red from right to left and/or from bottom to top.

1,333,330 in Egyptian hieroglyphs from the Edfu Temple (237–57 BCE) in Egypt.

The Babylonian system (form about 2000 BC) used only two symbols to write any number between 1 and 60, which shows that it was based on 60 and had some 10 biased system inside:




Also, it was a position value system, for example, number 62 was shown as  , which means one time 60 and 2. The use of this sexagesimal (60-based) system is still noticeable now, we have 60 minutes in one hour, 60 seconds in a minute, 360° for the full turn around.

Babylonian mathematical tablet Plimpton 322.Credit...Christine Proust and Columbia University

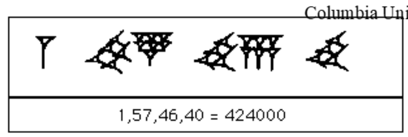
Now we adopt a notation where we separate the numerals by commas when we use our digits to represent Babylonian notation, for example, 1,57,46,40 represents the sexagesimal number

$$60^3 \cdot 1 + 60^2 \cdot 57 + 60^1 \cdot 46 + 60^0 \cdot 40$$

which, in decimal notation is 424000.

Here is **1,57,46,40 in Babylonian numerals**

https://mathshistory.st-andrews.ac.uk/HistTopics/Babylonian_mathematics/



Another very well-known numeral system is roman; it was used for thousands of years and in some cases is still used now as well. It's a “decimal”, 10 based system, but the symbols (letters) are used in an unusual way.

| Symbol | I | V | X | L | C | D | M |
|---|---|---|---|---|---|---|---|
| Value | 1 | 5 | 10 | 50 | 100 | 500 | 1000 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| I | II | III | IV | V | VI | VII | VIII | IX | X |

For example, 4 is one less than 5, so 4 can be written as IV. Same principle of subtractive notation is used for 9 -> IX, 40 and 90 -> XL and XC, 400 and 900 -> CD and CM
Some other examples:

- 29 = XX + IX = **XXIX**.
- 347 = CCC + XL + VII = **CCCXLVII**.
- 789 = DCC + LXXX + IX = **DCCLXXXIX**.
- 2,421 = MM + CD + XX + I = **MMCDXXI**

Any missing place (represented by a zero in the place-value equivalent) is omitted, as in Latin (and English) speech:

- 160 = C + LX = **CLX**
- 207 = CC + VII = **CCVII**
- 1,009 = M + IX = **MIX**
- 1,066 = M + LX + VI = **MLXVI**

Can non-decimal place-value system be created? For example, with base 5?
Let see, how we can create this kind of system (we use our normal digits).

| $Num_{10}$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $Num_5$ | 1 | 2 | 3 | 4 | 10 | 11 | 12 | 13 | 14 | 20 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
| 21 | 22 | 23 | 24 | 30 | 31 | 32 | 33 | 34 | 40 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 |
| 41 | 42 | 43 | 44 | 100 | 101 | 102 | 103 | 104 | 105 |

We only have 5 digits (0, 1, 2, 3, 4), and 4 first "natural" numbers in such system will be represented as one-digit numbers. Number 5 then should be shown as a 2-digit number, with fist digit 1 (place - value equal to $5^1$) and 0 of "units". Any number is now written in the form

$$\ldots + 5^3 \cdot n + 5^2 \cdot m + 5^1 \cdot k + 5^0 \cdot p, \qquad n, m, k, p \quad \text{are} \quad 0, 1, 2, 3, 4$$

$33 = 25 + 5 + 3 = 5^2 \cdot 1 + 5^1 \cdot 1 + 5^0 \cdot 3 = 113_5$

$195 = 125 + 25 \cdot 2 + 5 \cdot 4 = 5^3 \cdot 1 + 5^2 \cdot 2 + 5^1 \cdot 4 + 0 = 1240_5$

And vice versa, we can transform the number form 5-base to decimal system:

$2312_5 = 5^3 \cdot 2 + 5^2 \cdot 3 + 5^1 \cdot 1 + 2 = 250 + 75 + 5 + 2 = 332$

Let's do the addition of $1240_5$ and $2312_5$ (I omitted $_5$ notation):

$$\begin{array}{r} 1240 \\ +\ \underline{2312} \\ 4102 \end{array}$$

$4102_5 = 5^3 \cdot 4 + 5^2 \cdot 1 + 5^1 \cdot 0 + 2 = 125 \cdot 4 + 25 \cdot 1 + 0 + 2 = 500 + 25 + 2 = 527$

Let's try to introduce a new digit S for 10 and then create an 11 based system.

$11^2 = 121, \qquad 11^3 = 1331$

$890 = 121 \cdot 7 + 11 \cdot 3 + 10 = 11^2 \cdot 7 + 11^1 \cdot 3 + 10 = 73S_{11}$

$4S2_{11} = 11^2 \cdot 4 + 11^1 \cdot 10 + 2 = 484 + 110 + 2 = 596$

There is another very important place-value system: binary system, base 2 system where only two digits exist; 0, and 1.

| $Num_{10}$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| $Num_2$ | 1 | 10 | 11 | 100 | 101 | 110 | 111 | 1000 | 1001 | 1010 |

In this system place value of a digit is a power of 2:

$$\ldots + 2^3 \cdot (0,1) + 2^2 \cdot (0,1) + 2^1 \cdot (0,1) + 2^0 \cdot (0,1)$$

$11 = 8 + 2 + 1 = 2^3 \cdot 1 + 2^2 \cdot 0 + 2^1 \cdot 1 + 2^0 \cdot 1 = 1011_2$

$75 = 64 + 8 + 2 + 1 = 2^6 \cdot 1 + 2^5 \cdot 0 + 2^4 \cdot 0 + 2^3 \cdot 1 + 2^2 \cdot 0 + 2^1 \cdot 1 + 2^0 \cdot 1 = 1001011_2$

$$\begin{aligned} 189 &= 128 + 32 + 16 + 8 + 4 + 1 \\ &= 2^7 \cdot 1 + 2^6 \cdot 0 + 2^5 \cdot 1 + 2^4 \cdot 1 + 2^3 \cdot 1 + 2^2 \cdot 1 + 2^1 \cdot 0 + 2^0 \cdot 1 \\ &= 10111101_2 \end{aligned}$$

Binary numbers can be transferred to decimals too. We will do it from right to left for convenience.

$$11010111_2 = 2^0 \cdot 1 + 2^1 \cdot 1 + 2^2 \cdot 1 + 2^3 \cdot 0 + 2^4 \cdot 1 + 2^5 \cdot 0 + 2^6 \cdot 1 + 2^7 \cdot 1$$
$$= 128 + 64 + 16 + 4 + 2 + 1 = 215$$

**Exercises**:

1. Prove that
   $8^5 + 2^{11}$ is divisible by 17, $9^7 - 3^{10}$ is divisible by 20
   $16^4 + 2^{12}$ is divisible by 17, $4^3 - 2^4$ is divisible by 20

2. It is known that $a + 1$ is divisible by 3. Prove that $4 + 7a$ is divisible by 3 as well.
3. You are offered a job, which lasts for **7 weeks**. You get to choose your salary.

   - **Either**, you get \$100 for the first day, \$200 for the second day, \$300 for the third day. Each day you are paid \$100 more than the day before.

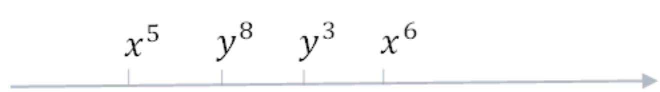
   - **Or**, you get 1 cent for the first day, 2 cents for the second day, 4 cents for the third day. Each day you are paid double what you were paid the day before.

   Which one will you prefer?

4. Write as a power:
   Example:
   $\frac{1}{2} = 2^{-1}$; $\frac{1}{4} = \frac{1}{2^2} = 2^{-2}$

   $\frac{1}{3}$; $\frac{1}{25}$; $\frac{1}{27}$; $\frac{1}{125}$;

5. $x^5 < y^8 < y^3 < x^6$
   Where 0 should be placed?




6. Write numbers 51 and 175 in binary system.

7. Write the numbers, written in the binary system in decimal system:

   $a. 11011011_2$; $b. 10001101_2$, $c. 111111111_2$
8. Write the numbers 245 and 324 in 6-based place-value system. Remember, that in this system you will have only 0, 1, 2, 3. 4, and 5 as digits.

9. Write the numbers $234_6$ and $403_6$ written in the 6-based place-value system (small number 6 shows that the number is not in decimal, but in 6-based system) in decimal system.