Classwork 25.

Combinatorics.

There are many tasks when you need to count the number of possible outcomes. For example, there are 5 chairs and 5 kids in the room. In how many ways can kids sit on these chairs? The first kid can choose any chair. The second kid can choose any of the 4 remaining chairs, the third has a choice between the three chairs, and the fifth kid has no choice at all. Therefore, there are $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ ways how all of them can choose their places. Thus, obtained long expression, $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$, can be written as 5!. By definition:

$$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5! \qquad or \quad n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1 = n!$$

This number 5! (or n!) shows the quantity of possible arrangement of 5 (n) objects, and is called permutations.

$$P = n!$$

Example 1:

1. *There are 10 books on the library shelf. How many different ways are there to place all these books on a shelf?*

Example 2:

2. *There are 10 books on the library shelf. 8 of them are authored by different authors and 2 are from the same author. How many different ways are there to place all these books on a shelf so that 2 books of one author will be next to each other?*

Because we want the two books of the same author be placed together, we can consider them as a single object and count the number of possible arrangements for 9 books, which is 9!. But in reality, for each of these arrangements, two books authored by the same author can be switch, so there are twice as many possible arrangements, $2 \cdot 9!$.

Now let's take a look on following problem:
There are 20 desks in our class and only 9 students. How many different ways are there to sit in the math class? How long it will take to try all of them, if you need 1second to switch places.
First student who came in the class has 20 desks to choose the place. Second student will have only 19 choices, and so on.
The total number of possible ways to sit is

$$20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 \cdot 13 \cdot 12 = 20 \cdot (20-1) \cdot \ldots \cdot (20-9+1) = 60949324800$$

It will take 42325920 days (or almost 116000 years) to try all of them.

In a general form we can ask about how many ways exists to arrange sample of m objects chosen out of n objects? The first object can be chosen by n different ways, second one by $(n-1)$ different way, and so on. The total number of ways will be

$$P(n,m) = n(n-1) \cdot \ldots \cdot (n-m+1)$$

This is the number of permutations of m objects chosen from n.
Can we write this formula in a shorter way?

$$P(n,m) = n(n-1)\cdot\ldots\cdot(n-m+1) = \frac{n(n-1)\cdot\ldots\cdot(n-m+1)\cdot(n-m)\cdot\ldots\cdot3\cdot2\cdot1}{(n-m)\cdot\ldots\cdot3\cdot2\cdot1} = \frac{n!}{(n-m)!}$$

If $m = n$, as in the first example, the formula is becoming

$$P(n,n) = n = \frac{n!}{(n-n)!} = n!$$

and it is clear that $0! = 1$, for everything to be consistent.

Example 3:

3. *In how many different ways the first three places can be awarded, if 20 people participated in the competition? In this case the repetition is not allowed, same person can't be placed in first and second place.*

Example 4:

4. *Peter has 5 final exams, LA, Math, Science, Social Studies, and Art. He can get A, B, C, and D as grades. How many different ways are there for his report card to look like? In this case we have to arrange 4 different grades in groups of 5 exams. The result of the exam can be any grade, even the same as the grade he got on another exam. For the first exam he can get any of the grade, so there is a choice of 4 grades for the first exam, as well as for any other exam.*

$$4\cdot4\cdot4\cdot4\cdot4 = 4^5$$

5. You have 5-digit lock and you forgot your code. You can check possible combinations with the speed 5 combination a minute. How long it will take you to open your locker?

Let's go back to the problem number 3 and compare it with the following problem:

6. *How many different ways are there to create a team of 3 students out of 20 students of math class to take a participation in the math Olympiad. What is similar and what is different between these two problems? The solution of the first one is*

$$P(20,3) = \frac{20!}{(20-3)!} = 20\cdot19\cdot18 = 6840$$

Does it matter, Alice is on the first place and Robert is on the second, or vice versa? Yes, it is a big difference for them. So, the group of three winners Alice, Robert, and Lia is different from the group Robert, Alice and Lia.

If we decide to solve the problem 6 the same way:

$$\frac{20!}{(20-3)!} = 20\cdot19\cdot18$$

We definitely will get the group of students (Alice, Robert, and Lia) and (Robert, Alice and Lia) as different arrangements. Does it matter for the group of three students who is going to participate in the math Olympiad? Each group of three will be counted more times than needed.

How many more times? Each group has $3 \cdot 2 \cdot 1 = 3! = 6$ different ways to be arranged, so we have to divide our result by this:

$$C(20,3) = \frac{20!}{(20-3)!} : 3! = \frac{20!}{(20-3)! \cdot 3!} = \frac{20 \cdot 19 \cdot 18}{6} = 1140$$

This type of choosing groups of three out of 20 and order doesn't matter, is called combinations, $C(20,3)$. Also $20C3$, or $\binom{20}{3}$ notations can be used.

In a general way, if we want to choose set of m objects out on n objects, regardless of order

$$C(n,m) = \binom{n}{m} = \frac{n!}{m!\,(n-m)!}$$

Probability.

What will you get if you toss a coin? Obviously, there will be either head or tail. If we would toss this coin many times, how many heads and how many tails we will register? The ratio of the desired outcome to the total number of possible outcomes is the probability of desired outcome to happen. In the example of a tossed coin there are two possible outcomes, head and tail, so the probability to get a head is (if it's a fair coin)

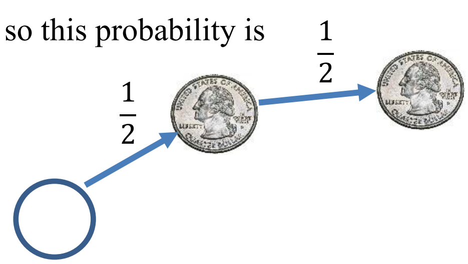
$$1 \; to \; 2, \quad or \; \frac{1}{2}; \quad or \; 0.5, \quad or \quad 50\%.$$

It doesn't mean that if you flip the coin twice you will definitely get a head.
Rather if you toss the coin 1000 times the head will appear about 500 times. More tossing, the closer the ratio is to ½. Let's check it!

If we roll a die (dice can be used as singular or plural, die is used only as singular), there are 6 possible outcomes, 1, 2, 3, 4, 5, and 6.
The probability to get 1 is $\frac{1}{6}$ (there is only 1 way to get desirable outcome and 6 possible outcomes.

$$\text{Probability of an event happening} = \frac{\text{Number of ways it can happen}}{\text{Total number of outcomes}}$$

What is a probability to get an even number on a die?
There are 3 possible ways to get even: 2, 4, 6. And 6 total outcomes, so this probability is $3:6 = 0.5$
Let's toss a coin twice. What is a probability to get head both times?



We can look at this event (get head twice) in two different ways:
First:
Probability to get a head first is ½. The probability to get second head is also ½. The probability to get two heads in a row is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$
Second:
There are 4 possible outcomes for two tosses:
HH, HT, TH, TT and only one (HH) possibility for us to get HH.

What is a chance to have a jackpot ticket in a lottery of 6/49? In another words, what is a probability that 6 numbers, chosen by the lottery will appear on the thicket?
The probability of the event is a ratio of numbers of the ways this event can happen to the total number of possible outcomes. In the case of lottery, there is only one winning set of six numbers. How many possible outcomes are there? We already know how to calculate it:

First number we can chose from 49, second number from 48 and so on.
To calculate the number of permutations $6P49 = 49 \cdot 48 \cdot 47 \cdot 46 \cdot 45 \cdot 44$
General calculation of permutation

$$P(n,m) = n(n-1)(n-2) \cdot \ldots \cdot (n-m+1)$$

In our lottery example last factor of the expression for permutation is $44 = 49 - 6 + 1$.

$$P(n,m) = n(n-1)(n-2) \cdot \ldots \cdot (n-m+1) =$$

$$\frac{n(n-1)(n-2) \cdot \ldots \cdot (n-m+1) \cdot (n-m)(n-m-1) \cdot \ldots \cdot 3 \cdot 2 \cdot 1}{(n-m)(n-m-1) \cdot \ldots \cdot 3 \cdot 2 \cdot 1} = \frac{n!}{(n-m)!}$$

$$6P49 = \frac{49!}{(49-6)!}$$

Order of the numbers in the set is not important, so we have to divide the number of permutations (order matter) by the number of permutations inside the group of 6. There are exactly 6! possible way to rearrange 6 objects

$$P(6.6) = 6 \cdot 5 \cdot \ldots \cdot 2 \cdot 1 = \frac{6!}{(6-6)!}$$

Division by 0! Mathematicians defined 0! as 1 .

$$P(6.6) = 6 \cdot 5 \cdot \ldots \cdot 2 \cdot 1 = \frac{6!}{(6-6)!} = 6!$$

Number of possible combinations of 6 numbers from49 is

$$C(49,6) = \binom{49}{6} = \frac{P(49,6)}{6!} = \frac{49!}{(49-6)!\,6!} = \frac{49 \cdot 48 \cdot \ldots \cdot 3 \cdot 2 \cdot 1}{43 \cdot 42 \cdot 41 \cdot \ldots \cdot 3 \cdot 2 \cdot 1 \cdot 6 \cdot 5 \cdot \ldots \cdot 2 \cdot 1}$$

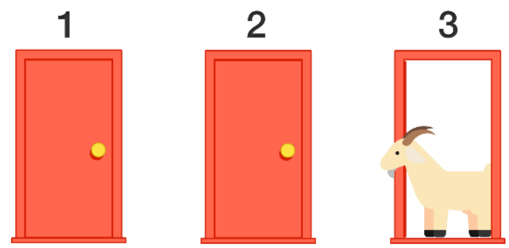
$$= \frac{49 \cdot 48 \cdot \ldots \cdot 44}{6 \cdot 5 \cdot \ldots \cdot 2 \cdot 1} = \frac{49 \cdot 8 \cdot 47 \cdot 23 \cdot 3 \cdot 11}{1} = 13{,}983{,}816$$

Number of combinations

$$C(n,m) = \binom{n}{m} = \frac{P(n,m)}{m!} = \frac{n!}{(n-m)!\,m!}$$

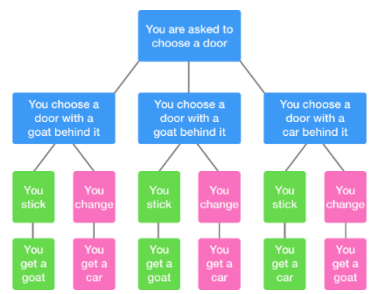
Probability to win a jackpot is $\frac{1}{13{,}983{,}816} \approx 7.15 \cdot 10^{-8}$

Monty Hall problem



In the problem, you are on a game show, being asked to choose between three doors. Behind each door, there is either a car or a goat. You choose a door. The host, Monty Hall, picks one of the other doors, which he knows has a goat behind it, and opens it, showing you the goat. (You know, by the rules of the game, that Monty will always reveal a goat.) Monty then asks whether you would like to switch your choice of door to the other remaining door. Assuming you prefer having a car more than having a goat, do you choose to switch or not to switch?

1. The host must always open a door that was not picked by the contestant.
2. The host must always open a door to reveal a goat and never the car.
3. The host must always offer the chance to switch between the originally chosen door and the remaining closed door.



There are three possible arrangements of one car and two goats behind three doors and the result of staying or switching after initially picking door 1 in each case

Behind door 1	Behind door 2	Behind door 3	Result if staying at door #1	Result if switching to the door offered
Goat	Goat	**Car**	Wins goat	**Wins car**
Goat	**Car**	Goat	Wins goat	**Wins car**
Car	Goat	Goat	**Wins car**	Wins goat

1. In a bag, there are three red marbles, two blue marbles and one yellow marble, find the probability of getting

   a. red marble
   b. blue marble
   c. yellow marble
2. There are 20 students in a class, each with a different first name (there are no students with the same first name). They are all very good at math, so they decided to randomly choose a team of three to go to the math Olympiad. What is the probability that Robert, John, and Mary will be on the same team?
3. There are three boxes, each containing balls numbered from 0 to 9. One ball is taken out of each box. What is the probability that: a) three ones are drawn; b) three equal numbers are drawn?
4. A two-digit number is written at random. What is the probability that the sum of the digits of this number is equal to 5?
5. Three tired cowboys walked into a saloon and hung their hats on the bison horn at the entrance. When they left in the deep of the night, they were unable to distinguish one hat from another, so they each randomly picked a hat. Find the probability that none of them took their own hat.
6. Alex wants to see how many times a "double" comes up when throwing 2 dice. After 100 trials, Alex has 19 "double" events ... is that close to what you would expect?
7. Throw a dice 3 times. What's the probability that we have three 5?
8. From a pack of 52 cards, a card is drawn at random. What is the probability of getting a queen?

9. Throw 2 dices simultaneously. What is the probability that the summation of the numbers is multiply of 4?

10. 10 points are marked on the plane so that no three of them lie on the same (straight) line. How many segments are there with ends at these points? If there are 50 points? 100 points?

11. 10 points are marked on the plane so that no three of them lie on the same (straight) line. How many triangles are there with vertices at these points? If there are 50 points? 100 points?

12. There are 3 starters, 4 entrees, and 4 desserts in the price fix dinner. How many different ways are there to fix your diner?

13. How many different five-digit numbers, not containing identical digits, can be written using the digits 1, 2, 3, 4, 5?
14. How many different five-digit numbers, not containing identical digits, can be written using the digits 1, 2, 3, 4, 5, such that:

    a. The last digit is 3?

    b. The first digit is 2, and the last digit is 4?

15. Show that

    $(57! + 58!)$ is divisible by 59. (Hint: find common factors.)