Math 5b, classwork 2.

Exponent.

Exponentiation is a mathematical operation, of multiplication of the same number several times, written as a^n, involving two numbers, the base *a* and the exponent *n*.

$$a^n = \underbrace{a \cdot a \cdot a \dots \cdot a}_{n\ times}$$

In that case, a^n is called the *n*-th power of *a*, or *a* raised to the power *n*.

The exponent indicates how many copies of the base are multiplied together.

For example, $3^5 = 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 = 243$. The base 3 appears 5 times in the repeated multiplication, because the exponent is 5. Here, 3 is the *base*, 5 is the *exponent*, and 243 is the *power* or, more specifically, *the fifth power of 3*, *3 raised to the fifth power*, or *3 to the power of 5*.

Properties of exponent:

- To multiply two powers with the same base we can write:

$$3^{10} \cdot 3^{15} = \underbrace{3 \cdot 3 \dots \cdot 3}_{10\ times} \cdot \underbrace{3 \cdot 3 \dots \cdot 3}_{15\ times} = \underbrace{3 \cdot 3 \cdot 3 \dots \cdot 3}_{10+15\ times} = 3^{10+15} = 3^{35}$$

In general form the same expression should be written as following:

$$a^m \cdot a^n = \underbrace{a \cdot a \dots \cdot a}_{m\ times} \cdot \underbrace{a \cdot a \dots \cdot a}_{n\ times} = \underbrace{a \cdot a \cdot a \dots \cdot a}_{m+n\ times} = a^{m+n}$$

(for now, let assume that n and m are both natural numbers)

- To raise an exponent to another exponent we can write;

$$(3^{10})^{15} = \underbrace{3^{10} \cdot 3^{10} \cdot \dots \cdot 3^{10}}_{15\ times} = \underbrace{\underbrace{3 \cdot 3 \cdot \dots \cdot 3}_{10\ times} \cdot \dots \cdot \underbrace{3 \cdot 3 \cdot \dots \cdot 3}_{10\ times}}_{15\ times} = 3^{10 \cdot 15} = 3^{150}$$

General expression will look like this:

$$(a^m)^n = \underbrace{a^m \cdot a^m \cdot \dots \cdot a^m}_{n\ times} = \underbrace{\underbrace{a \cdot a \cdot \dots \cdot a}_{m\ times} \cdot \dots \cdot \underbrace{a \cdot a \cdot \dots \cdot a}_{m\ times}}_{m\ times} = a^{m \cdot n}$$

- By multiplying an exponent by one more base we are going to get the expression:

$$3^{10} \cdot 3 = \underbrace{3 \cdot 3 \cdot 3 \dots \cdot 3}_{10\ times} \cdot 3 = \underbrace{3 \cdot 3 \cdot 3 \cdot 3 \dots \cdot 3}_{10+1\ times} = 3^{10+1} = 3^{10} \cdot 3^1$$

As you can see, number can be considered as a number in the first power, and this assumption is very well aligned with first two properties of the operation of raising to a power:

$$3^{10} \cdot 3 = 3^{10+1} = 3^{10} \cdot 3^1$$

and

$$(3^{10})^1 = 3^{10 \cdot 1} = 3^{10}$$

General way to write this is

$$a^n \cdot a = \underbrace{a \cdot a \cdot a \ldots \cdot a}_{n\ times} \cdot a = \underbrace{a \cdot a \cdot a \cdot a \ldots \cdot a}_{n+1\ times} = a^{n+1} = a^n \cdot a^1$$

1. $a^n = \underbrace{a \cdot a \cdot a \ldots \cdot a}_{n\ times}$
2. $a^n \cdot a^m = a^{n+m}$
3. $(a^n)^m = a^{n \cdot m}$
4. $a^1 = a$, for any a
5. $a^0 = 1$, for any $a \neq 0$
6. $(a \cdot b)^n = a^n \cdot b^n$

Any number in the first power is equal to itself.

In order to have the set of power properties consistent, $a^1 = a$ for any number *a*.

Next step is a little more sophisticated.

- Let's multiply an exponent by 1. We always can multiply by 1, by identity property of multiplication, nothing will change. But now we know the first property of power and can do this:

$$3^{10} \cdot 1 = 3^{10} = 3^{10+0} = 3^{10} \cdot 3^0$$

Or, in general form:

$$a^n = a^n \cdot 1 = a^{n+0} = a^n \cdot a^0$$

In order to have the set of properties of exponent consistent, $a^0 = 1$ for any number *a,* but 0 (for any $a \neq 0$)

- For two numbers, raised to the same power we can write

$$(2 \cdot 3)^{10} = \underbrace{(2 \cdot 3) \cdot \ldots \cdot (2 \cdot 3)}_{10\ times} = \underbrace{2 \cdot \ldots \cdot 2}_{10\ times} \cdot \underbrace{3 \cdot \ldots \cdot 3}_{10\ times} = 2^{10} \cdot 3^{10}$$

$$(a \cdot b)^n = \underbrace{(a \cdot b) \cdot \ldots \cdot (a \cdot b)}_{n\ times} = \underbrace{a \cdot \ldots \cdot a}_{n\ times} \cdot \underbrace{b \cdot \ldots \cdot b}_{n\ times} = a^{10} \cdot b^{10}$$

- A positive number raised into any power will result a positive number.
- A negative number, raised in a power, represented by an even number is positive, represented by an odd number is negative.

Exponents can help write numbers in their expanded form. In our base-10 place value system, the position of the digit indicates which power of 10 should be multiplied by the digit. For example:

$$345 = 300 + 40 + 5 = 100 \cdot 3 + 10 \cdot 4 + 1 \cdot 5 = 10^2 \cdot 3 + 10^1 \cdot 4 + 10^0 \cdot 5$$

Exponent is very interesting mathematical operation. There is the story of the invention of the game of chess. The king ordered a new game because he was bored by the old games, was so happy about the new chess game that he said to the inventor: "*Name your reward and you will get it!*" The inventor asked for a simple reward. *"I would like to have one grain of rice on the first chess square, two on the second, four on the third and so on, doubling the amount of rice every square.*" The legend says that the King was surprised he didn't ask for gold but was quite content that the inventor asked for so little. But when the court scholars told him there wasn't enough rice in the whole world to fill the chess board, he had to admit his loss:

$$1 + 2 + 2^2 + 2^4 + \cdots + 2^{63} = 18{,}446{,}744{,}073{,}709{,}551{,}615$$

The weight of the rice grain is about 0.03g. so:

$$1.8 \cdot 10^{19} \cdot 0.03 = 5.4 \cdot 10^{17}\ g.\ or\ about\ 5.4 \cdot 10^{14} kg\ or\ 10^{15} lb.$$

Another example:

If I put 1000 dollars into the bank account and will get 5% annually (I will not take out anything).

In 1 year, my deposit will become

$$1.05 \cdot 1000 = 1050$$

Next year:

$$1.05 \cdot 1.05 \cdot 1000 = 1.05^2 \cdot 1000 = 1102.5$$

In 100 years:

$$1.05^{100} \cdot 1000 \approx 131.5 \cdot 1000 = 131500$$

Exercises:

1. What should be the exponent for the equation to hold?

   $a.\ \ 8^* = 512;\quad b.\ \ 2^* = 64;\quad c.\ \ 3^* = 81;\quad d.\ \ 7^* = 343$

2. What digits should be put instead of * to get true equality? How many solutions does each problem have?

   $a.\ (2\,*)^2 = **\,1;\quad b.\ (3\,*)^2 = ***\,6$

   $c.\ (7\,*)^2 = ***\,5\quad d.\ (2\,*)^2 = **\,9,\quad e.\ \ (3\,*)^2 = **\,1$

3. In a magical lake, the number of water lilies doubles every night. On March 1st, the magician planted the first lily, and in 90 nights, the entire lake was covered with lilies. On which day was only half of the lake covered?

4. Evaluate:

   $(-3)^2;\quad -3^2;\quad (-3)^3;\quad 2^7;\quad (-2)^7;\quad -2^7;\quad (2 \cdot 3)^3;\quad 2 \cdot 3^3;\quad \left(\frac{1}{3}\right)^2;\quad \frac{1}{3^2};$

5. Represent numbers as a power of 10:

   Example: $1000^3 = (10^3)^3 = 10^{3 \cdot 3} = 10^9$

   $100^2;\quad 100^3;\quad 100^4;\quad 100^5;\quad 100^6;$

6. Write the numbers in their extended form;

   231, 305, 2001, 2931, 3400.

7. Write the number which extended form is written below;

   Example: $2 \cdot 10^3 + 7 \cdot 10^2 + 2 \cdot 10^1 + 6 \cdot 10^0 = 2726;$

   $a.\ 3 \cdot 10^3 + 5 \cdot 10^2 + 2 \cdot 10^1 + 1 \cdot 10^0;\qquad b.\ 6 \cdot 10^3 + 1 \cdot 10^2 + 0 \cdot 10^1 + 3 \cdot 10^0;$

   $c.\ 8 \cdot 10^3 + 5 \cdot 10^1 + 3 \cdot 10^0;\qquad e.\ 3 \cdot 10^3 + 2 \cdot 10^2 + 5 \cdot 10^1 + 4 \cdot 10^0;$

8. Compare:

$a.\ 5^3$ *and* $5 \cdot 3$; $b.\ 12^2$ *and* $12 \cdot 2$; $c.\ 2^5$ *and* 5^2;

$d.\ 3^4$ *and* 4^3; $e.\ 2^2$ *and* $2 \cdot 2$; $f.\ 2^4$ *and* 4^2;

9. Which number is greater?

$a.\ 29^2$ *or* 1000; $b.\ 43^2$ *or* 1500; $c.\ 49^2$ *or* 3000; $d.\ 67^2$ *or* 3500

10. Write the following expressions in a shorter way replacing product with power:

Examples:

$(-a) \cdot (-a) \cdot (-a) \cdot (-a) = (-a)^4$, $3m \cdot m \cdot m \cdot 2k \cdot k \cdot k \cdot k = 6m^3k^4$

$a.\ (-y) \cdot (-y) \cdot (-y) \cdot (-y)$; $b.\ (-5m)(-5m) \cdot 2n \cdot 2n \cdot 2n$;

$c.\ -y \cdot y \cdot y \cdot y$; $d.\ -5m \cdot m \cdot 2n \cdot n \cdot n$;

$e.\ (ab) \cdot (ab) \cdot (ab) \cdot (ab) \cdot (ab) \cdot (ab)$; $f.\ p - q \cdot q \cdot q \cdot q \cdot q$;

$g.\ a \cdot b \cdot b \cdot b \cdot b \cdot b$; $h.\ (p-q) \cdot (p-q) \cdot (p-q)$;

11. Simplify the expressions:

$a.\ 2^4 + 2^4$; $b.\ 2^m + 2^m$; $c.\ 2^m \cdot 2^m$;

$d.\ 3^2 + 3^2 + 3^2$; $e.\ 3^k + 3^k + 3^k$; $f.\ 3^k \cdot 3^k \cdot 3^k$;

12. What will be last digit of

$a.\ 2^{22}$; $b.\ 3^{33}$; $c.\ 4^{44}$; $d.\ 5^{55}$; $e.\ 6^{66}$; $f.\ 7^{77}$;

13. Compare:

a. 127^{23} and 513^{18} b. 9997^{10} and 100003^8 c. 5^{300} and 3^{500}

14. Write the following numbers as the power of base 10:

10, 100, 1000, 10000, 100000, 1000000

0.1, 0.01, 0.001, 0.0001, 0.00001, 0.000001

15. Reduce the fractions

$a.\ \frac{49^4 \cdot 7^5}{7^{12}}$; $b.\ \frac{3^{10} \cdot 27}{81^3}$; $c.\ \frac{125^3 \cdot 5^7}{5^{18}}$;