MATH 5: HANDOUT 26

DE MORGAN'S LAWS, INCLUSION-EXCLUSION, AND PIGEONHOLE PRINCIPLE

Introduction

In the previous handout you met the basic operations of set theory: union, intersection, complement, subset. In this handout we put those tools to serious work. We'll meet three powerful ideas that all answer the same general question: *how do we count things cleverly, without listing them one by one?*

- **De Morgan's Laws** tell us how the operations $\cup$, $\cap$, and "complement" interact — they are the basic algebra of sets.
- **Inclusion-Exclusion** gives a precise formula for counting elements in a union, even when the sets overlap.
- **The Pigeonhole Principle** lets us prove that something must exist (a duplicate, a coincidence, a special pair) without finding it explicitly.

These three ideas are simple to state but turn up everywhere — in computer science, statistics, logic, probability, and pure mathematics.

De Morgan's Laws

There is a beautiful symmetry between complements and the operations $\cup$ and $\cap$. Augustus De Morgan (1806–1871) discovered two elegant laws that describe it. Informally: *when you take a complement, union and intersection swap*.

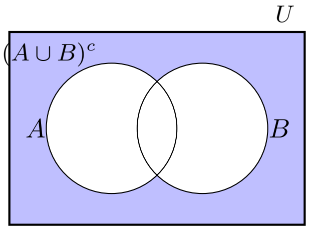
Theorem

De Morgan's Laws. For any sets A and B within a universal set U:

$$(A \cup B)^c = A^c \cap B^c \qquad \text{and} \qquad (A \cap B)^c = A^c \cup B^c.$$

In words: the complement of a union is the intersection of the complements, and vice versa.

To see why the first law holds: an element x is in $(A \cup B)^c$ exactly when it is *not* in $A \cup B$ — meaning it is not in A *and* not in B — which is exactly $A^c \cap B^c$.



$= A^c \cap B^c$

Example 1. Let $U = \{1,2,3,4,5,6\}$, $A = \{1,2,3\}$, $B = \{3,4,5\}$. Verify the first law.
Left side: $A \cup B = \{1,2,3,4,5\}$, so $(A \cup B)^c = \{6\}$.
Right side: $A^c = \{4,5,6\}$, $B^c = \{1,2,6\}$, so $A^c \cap B^c = \{6\}$. ✓

Three sets and beyond. The same pattern continues for any number of sets. For three sets:

$$(A \cup B \cup C)^c = A^c \cap B^c \cap C^c, \qquad (A \cap B \cap C)^c = A^c \cup B^c \cup C^c.$$

The rule never changes: *when you take a complement, every* $\cup$ *becomes* $\cap$ *and every* $\cap$ *becomes* $\cup$.

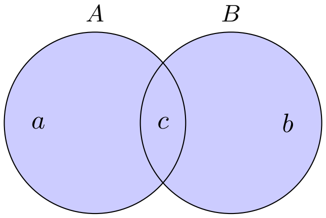
Quick Check

1. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{1, 3, 5, 7\}$, $B = \{3, 4, 5, 6\}$. Verify the first De Morgan law: $(A \cup B)^c = A^c \cap B^c$.

Counting with Sets: Inclusion-Exclusion

One of the most powerful applications of sets is counting. To count $|A \cup B|$, can we just add $|A| + |B|$? Not quite — let us see exactly why, step by step.

Every Venn diagram of two overlapping sets has three regions. Label them: a = elements *only in* A, c = elements *in both*, b = elements *only in* B.



From the diagram: $|A| = a + c$, $|B| = b + c$, $|A \cap B| = c$, and we want $|A \cup B| = a + b + c$.

What happens if we just add $|A| + |B|$?

$$|A| + |B| = (a + c) + (b + c) = a + b + \mathbf{2c}.$$

The middle region c got counted **twice**. Fix: subtract $|A \cap B| = c$ once.

Theorem

Inclusion-Exclusion (Two Sets). $|A \cup B| = |A| + |B| - |A \cap B|$

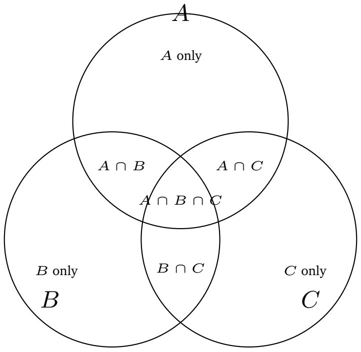
Example 2. In a class of 30 students, 18 play soccer and 15 play basketball, 10 play both. How many play at least one?

$$|S \cup B| = 18 + 15 - 10 = 23. \quad \text{The remaining 7 play neither.}$$

Quick Check

2. In a class of 40 students, 25 play chess and 20 play checkers. If 10 play both, how many play at least one game?

The same idea extends to three sets. A three-set Venn diagram has **seven** regions:



Region	$+\lvert A\rvert+\lvert B\rvert+\lvert C\rvert$	$-\lvert A\cap B\rvert-\cdots$	$+\lvert A\cap B\cap C\rvert$	Total
Only in one set	$+1$	0	0	1 ✓
In exactly two sets	$+2$	-1	0	1 ✓
In all three sets	$+3$	-3	$+1$	1 ✓

Theorem

Inclusion-Exclusion (Three Sets).

$$|A\cup B\cup C| = |A|+|B|+|C|-|A\cap B|-|A\cap C|-|B\cap C|+|A\cap B\cap C|$$

Example 3. In a group of 40 students: 22 like math, 18 like science, 20 like art, 12 like math and science, 10 like math and art, 8 like science and art, 5 like all three.

$$|M\cup S\cup A| = 22+18+20-12-10-8+5 = 35.$$

So 35 students like at least one subject, and $40-35=5$ like none.

Quick Check

3. Of 60 people: 30 like tea, 25 like coffee, 20 like juice, 10 like tea and coffee, 8 like tea and juice, 7 like coffee and juice, 4 like all three. How many like at least one?

The Pigeonhole Principle

The Pigeonhole Principle is one of the simplest yet most powerful ideas in mathematics. It states something that seems obvious, but leads to surprising conclusions.

Theorem

Pigeonhole Principle (Basic Form). If n items are placed into k containers and $n>k$, then at least one container holds more than one item.

Example 4. If there are 13 people in a room, prove that at least two were born in the same month.

Solution: 13 pigeons (people), 12 pigeonholes (months). Since $13>12$, some month contains at least two people.

Example 5. A drawer has only black and white socks. How many must you pull out in the dark to guarantee a matching pair?

Solution: 2 colors (pigeonholes). Pull out 3 socks (pigeons): by Pigeonhole, at least two are the same color.

Quick Check

4. If a class has 367 students, can you guarantee that at least two share a birthday?
5. A box contains red, blue, and green marbles. How many must you draw to guarantee two of the same color?

Generalized Pigeonhole Principle

The basic principle tells us that *some* container holds at least 2 items — but often we can say much more. Suppose 50 students are distributed across 12 birth months. The basic principle only tells us some month is shared by at least 2 students. But with 50 students and only 12 months, surely some month must be much more crowded than that! How crowded, exactly? If we spread 50 students as evenly as possible, each month would get about $50/12 \approx 4.17$ students. Since we can't have a fraction of a student, at least one month must have $\lceil 4.17 \rceil = 5$. The generalized principle makes this precise for any numbers.

Theorem

Generalized Pigeonhole Principle. If n items are placed into k containers, then at least one container holds at least $\lceil n/k \rceil$ items, where $\lceil x \rceil$ is the **ceiling** of x: the smallest integer $\geq x$.

Why does this work? Suppose for contradiction that every container holds *at most* $\lceil n/k \rceil - 1$ items. Then the total number of items is at most

$$k \cdot \big(\lceil n/k \rceil - 1\big) < k \cdot \frac{n}{k} = n.$$

But we have n items — contradiction. So some container must hold at least $\lceil n/k \rceil$ items.

Example 6. In a group of 50 people, at least how many share a birth month?

$$\lceil 50/12 \rceil = \lceil 4.17\ldots \rceil = 5.$$

At least 5 people share a birth month.

Example 7. In a group of 30 students, at least how many were born in the same month? At least how many were born in the same season (spring, summer, fall, winter)?

$$\lceil 30/12 \rceil = \lceil 2.5 \rceil = 3 \text{ share a month.} \qquad \lceil 30/4 \rceil = \lceil 7.5 \rceil = 8 \text{ share a season.}$$

The reverse question. Sometimes we ask: *how many items do we need to guarantee at least m in one container?*

If there are k containers and we want to guarantee at least m in one, we need at least $k(m-1)+1$ items. (The worst case is $m-1$ in every container, giving $k(m-1)$ items with no container reaching m; the next item pushes one container to m.)

Example 8. A drawer has socks in 5 colors. How many socks must you pull out to guarantee:

- At least 2 of the same color? $5 \cdot (2-1) + 1 = 6$ socks.
- At least 4 of the same color? $5 \cdot (4-1) + 1 = 16$ socks.

Quick Check

6. In a school of 1000 students, at least how many must share the same birthday (day and month)?
7. If you draw 15 cards from a standard deck, at least how many must be of the same suit?

Clever Applications

The power of the principle comes from cleverly choosing what the "pigeons" and "holes" are.

Example 9. Hair Count. Human heads have at most about 150,000 hairs. New York City has 8 million people. Prove that at least two New Yorkers have exactly the same number of hairs.

Pigeons: 8,000,000 people. Holes: 150,001 possible counts (0 to 150,000). Since $8{,}000{,}000 \gg 150{,}001$, at least two people share the same count.

Example 10. Divisibility. Among any 5 integers, prove that some two have a difference divisible by 4.

When dividing by 4, remainders are 0, 1, 2, or 3 — four pigeonholes. With 5 integers, two share the same remainder, so their difference is divisible by 4.

Example 11. Multiples Made of 0s and 1s. For any positive integer n, prove that some multiple of n has a decimal representation containing only the digits 0 and 1.

Solution. Consider the $n+1$ numbers

$$1,\ 11,\ 111,\ 1{,}111,\ \ldots,\ \underbrace{11\ldots1}_{n+1\text{ ones}}.$$

Each leaves some remainder when divided by n, but there are only n possible remainders $(0, 1, \ldots, n-1)$. By Pigeonhole, two of these numbers — say A and B with $A > B$ — must share the same remainder. Then $A - B$ is divisible by n, and $A - B$ has the form

$$\underbrace{11\ldots1}_{\text{some 1s}}\underbrace{00\ldots0}_{\text{some 0s}},$$

exactly the kind of multiple we wanted.

Example. For $n = 7$, computing $1,\ 11,\ 111,\ 1{,}111, \ldots$ modulo 7 quickly turns up $111{,}111 = 7 \cdot 15{,}873$. So $111{,}111$ is a multiple of 7 made entirely of 1s — a fact you'd never guess by looking, but Pigeonhole guaranteed it had to exist.

Quick Check

8. Among any 6 integers, prove that some two have a difference divisible by 5.
9. In any group of 13 people, prove that at least two were born on the same day of the week.

An important feature: the Pigeonhole Principle proves something *exists* without telling us *which* example. We proved two New Yorkers share a hair count — but we don't know who they are.

Where These Ideas Show Up

The three tools you've just learned aren't isolated curiosities — they're foundational ideas that power a great deal of modern science and technology.

De Morgan's Laws in Logic and Programming

Computers run on **logical conditions**: "do this if X is true." De Morgan's laws translate directly into Boolean algebra:

$$\text{NOT}(P \text{ OR } Q) \ \equiv\ (\text{NOT } P) \text{ AND } (\text{NOT } Q),$$

which is the same as $(A \cup B)^c = A^c \cap B^c$ with sets replaced by truth values. Every programmer uses this when simplifying `if` statements; every digital circuit uses it when designing AND/OR/NOT gates. The laws are also the heart of how search engines parse queries like "**not** (cats **or** dogs)."

Inclusion-Exclusion in Surveys and Probability

Whenever a survey asks "how many people do A, B, and/or C," inclusion-exclusion is the only way to combine the answers without double-counting. Pollsters, marketers, and epidemiologists all use it constantly. It's also the backbone of probability: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ has the exact same shape as the IE formula and comes up every time you compute the chance that "at least one of several things happens."

Pigeonhole in Computer Science

Pigeonhole is everywhere in computer science:

- **Hashing.** A hash function maps any input (a string, a file, a password) to a fixed-size code, say 128 bits. There are infinitely many possible inputs, but only 2^{128} codes. By Pigeonhole, two different inputs must map to the same code — this is called a *hash collision*, and avoiding them is a major problem in cryptography.
- **Lossless compression.** You cannot compress *every* file to be shorter than itself. (Pigeonhole: too many possible files, not enough short codes.) So compression algorithms work by making common files short and rare files long.
- **Theorem proving.** "At least two of these objects must share a property" is the start of countless proofs in computer science, combinatorics, number theory, and graph theory.

The Big Picture

You might notice the underlying theme: all three tools handle situations where *listing every case is impossible*. We can't list 8 million New Yorkers' hair counts, or every 128-bit hash, or every voter in a national survey. What we can do is **reason about the structure** of the situation and reach a conclusion.

This is one of mathematics' superpowers: turning "we can't possibly check everything" into "we don't need to."

Key Takeaways

- De Morgan's Laws: $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$. Complementing swaps $\cup$ and $\cap$.
- Inclusion-Exclusion (2 sets): $|A \cup B| = |A| + |B| - |A \cap B|$.
- Inclusion-Exclusion (3 sets): add singles, subtract pairwise, add triple.
- Basic Pigeonhole: n items in k containers ($n > k$) $\Rightarrow$ some container has ≥ 2 items.
- Generalized Pigeonhole: at least one container has $\geq \lceil n/k \rceil$ items.
- The art of Pigeonhole is identifying what the pigeons and holes are.
- Pigeonhole proves existence — it does not find the specific example.

Common Mistakes

- **Double-counting in union.** Don't forget to subtract $|A \cap B|$!
- **Misapplying De Morgan's.** $(A \cup B)^c = A^c \cap B^c$, not $A^c \cup B^c$. The operation flips.
- **Wrong sign in 3-set IE.** You add $|A \cap B \cap C|$ back (not subtract it).
- **Off-by-one in Pigeonhole.** To guarantee 2 items in one hole among k holes, you need $k+1$ items, not k.
- **Wrong ceiling.** $\lceil 50/12 \rceil = 5$, not 4. Ceiling rounds *up*.

Classwork

1. Let $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $A = \{2, 4, 6, 8, 10\}$, $B = \{1, 2, 3, 4, 5\}$.
   (a) Find $(A \cup B)^c$ and $A^c \cap B^c$. Are they equal?
   (b) Find $(A \cap B)^c$ and $A^c \cup B^c$. Are they equal?
2. In a survey of 100 students: 60 like pizza, 45 like burgers, 25 like both.
   (a) How many like at least one?
   (b) How many like neither?
3. In a group of 50 students: 25 study Spanish, 20 study French, 10 study German, 8 study Spanish and French, 5 study Spanish and German, 4 study French and German, 2 study all three. How many study at least one language?
4. A drawer contains red, blue, green, and yellow socks (all mixed up). How many socks must you pull out in the dark to guarantee:
   (a) A matching pair?
   (b) Three socks of the same color?
5. In a class of 30 students, at least how many must be born in the same month?
6. Among any 10 integers, prove that at least two have the same remainder when divided by 9.

Classwork Solutions

1. (a) $A \cup B = \{1, 2, 3, 4, 5, 6, 8, 10\}$, so $(A \cup B)^c = \{7, 9\}$. $A^c = \{1, 3, 5, 7, 9\}$, $B^c = \{6, 7, 8, 9, 10\}$, so $A^c \cap B^c = \{7, 9\}$. ✓
   (b) $A \cap B = \{2, 4\}$, so $(A \cap B)^c = \{1, 3, 5, 6, 7, 8, 9, 10\}$. $A^c \cup B^c = \{1, 3, 5, 7, 9\} \cup \{6, 7, 8, 9, 10\} = \{1, 3, 5, 6, 7, 8, 9, 10\}$. ✓
2. $|P \cup B| = 60 + 45 - 25 = 80$. 80 like at least one; $100 - 80 = 20$ like neither.
3. $|S \cup F \cup G| = 25 + 20 + 10 - 8 - 5 - 4 + 2 = 40$ students.
4. (a) 4 colors = 4 holes. Need $4 + 1 = 5$ socks to guarantee a matching pair.
   (b) Worst case: 2 of each color = $2 \times 4 = 8$. Need 9 socks to guarantee 3 of one color.
5. $\lceil 30/12 \rceil = \lceil 2.5 \rceil = 3$ students share a birth month.
6. There are 9 possible remainders mod 9 (namely 0–8). With 10 integers and 9 remainders, at least two share a remainder.

Homework

De Morgan's Laws and Inclusion-Exclusion

1. Let $U = \{1, 2, \ldots, 12\}$, $A = \{2, 4, 6, 8, 10, 12\}$, $B = \{3, 6, 9, 12\}$.
   - (a) Find $(A \cup B)^c$ and $A^c \cap B^c$. Are they equal?
   - (b) Find $(A \cap B)^c$ and $A^c \cup B^c$. Are they equal?
2. Let A be the set of multiples of 4 less than 30, and B be the set of multiples of 6 less than 30.
   - (a) List the elements of A and B.
   - (b) Find $A \cap B$.
   - (c) Find $|A \cup B|$ using inclusion-exclusion.
3. In a group of 100 students: 40 play soccer, 50 play basketball, 30 play baseball, 20 play soccer and basketball, 15 play soccer and baseball, 10 play basketball and baseball, and 5 play all three.
   - (a) How many play at least one sport?
   - (b) How many play exactly two sports?
   - (c) How many play exactly one sport?
4. Prove De Morgan's second law $(A \cap B)^c = A^c \cup B^c$ by showing both inclusions.
   Hint: $x \in (A \cap B)^c$ means $x \notin A \cap B$, which means $x \notin A$ or $x \notin B$.

Pigeonhole Principle

5. A bag contains red, blue, green, yellow, and orange candies. How many must you pick to guarantee:
   - (a) Two of the same color?
   - (b) Four of the same color?
6. In a group of 50 people, at least how many must have been born:
   - (a) In the same month?
   - (b) On the same day of the week?
7. Among any 11 integers, prove that at least two have a difference divisible by 10.
8. In a standard deck of 52 cards, how many cards must you draw to guarantee:
   - (a) Two cards of the same suit?
   - (b) Two cards of the same rank (e.g., two Kings)?
   - (c) All four cards of some rank?
9. H Prove that in any group of 6 people, either 3 of them all know each other, or 3 of them are all strangers.
   Hint: Pick one person and consider their relationships with the other 5.
10. M Five points are placed inside a square of side length 2. Prove that at least two points are within distance $\sqrt{2}$ of each other.
    Hint: Divide the square into 4 smaller squares.
11. H **Lattice midpoints.**

(a) Five points are chosen in the plane, all with integer coordinates (these are called *lattice points*). Prove that the midpoint of some pair of them is also a lattice point.
Hint: sort each point by the parities of its x- and y-coordinates.

(b) Show that the bound of 5 is tight: find *four* lattice points such that no pair has an integer-coordinate midpoint.

12. H Among any 52 integers, prove you can find two whose sum or difference is divisible by 100.
Hint: Consider remainders mod 100, and pair up remainders that sum to 100.

Quick Check Answers

1. $A \cup B = \{1, 3, 4, 5, 6, 7\}$, so $(A \cup B)^c = \{2, 8\}$. $A^c = \{2, 4, 6, 8\}$, $B^c = \{1, 2, 7, 8\}$, so $A^c \cap B^c = \{2, 8\}$. ✓
2. $|C \cup Ch| = 25 + 20 - 10 = 35$ students.
3. $|T \cup C \cup J| = 30 + 25 + 20 - 10 - 8 - 7 + 4 = 54$ people.
4. Yes: $367 > 366$ (max days in a year), so at least two share a birthday.
5. 4 marbles (3 colors = 3 holes, need 4 to guarantee a repeat).
6. $\lceil 1000/366 \rceil = \lceil 2.73 \ldots \rceil = 3$ students share the same birthday.
7. $\lceil 15/4 \rceil = 4$ cards of the same suit.
8. 6 integers, 5 possible remainders mod 5. By Pigeonhole, two share a remainder, so their difference is divisible by 5.
9. 13 people, 7 days of the week. By Pigeonhole, at least $\lceil 13/7 \rceil = 2$ share a day.