MATH 5: HANDOUT 26

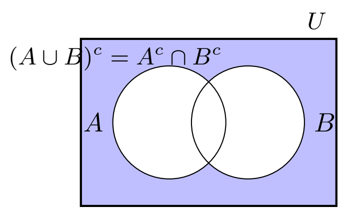
De Morgan's Laws, Inclusion-Exclusion, and Pigeonhole Principle (Summary)

De Morgan's Laws

Theorem 1 (De Morgan's Laws)**.** For any sets A and B within a universal set U:

$$(A \cup B)^c = A^c \cap B^c \qquad \text{and} \qquad (A \cap B)^c = A^c \cup B^c.$$

Why: $x \in (A \cup B)^c$ means $x \notin A$ *and* $x \notin B$, i.e. $x \in A^c \cap B^c$. Complementing a union/intersection **swaps** $\cup$ and $\cap$.



Inclusion-Exclusion

Simply adding $|A| + |B|$ double-counts the overlap. Fix by subtracting it once:

Theorem 2 (Inclusion-Exclusion — Two Sets)**.** $|A \cup B| = |A| + |B| - |A \cap B|$

For three sets, label the seven regions and track how many times each is counted:

Region	$+\lvert A\rvert + \lvert B\rvert + \lvert C\rvert$	$-\lvert A \cap B\rvert - \cdots$	$+\lvert A \cap B \cap C\rvert$	Total
Only in one set	$+1$	0	0	1 ✓
In exactly two sets	$+2$	-1	0	1 ✓
In all three sets	$+3$	-3	$+1$	1 ✓

Theorem 3 (Inclusion-Exclusion — Three Sets)**.**

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

The Pigeonhole Principle

Theorem 4 (Pigeonhole Principle)**.** If n items are placed into k containers and $n > k$, then at least one container holds **more than one** item.

Theorem 5 (Generalized Pigeonhole Principle)**.** If n items are placed into k containers, then at least one container holds at least $\lceil n/k \rceil$ items, where $\lceil x \rceil$ is the **ceiling** (smallest integer $\geq x$).

The art of the principle is choosing the right pigeons and holes. Pigeonhole proves *existence* without identifying *which* example.

Example 1. In a room of 13 people, at least two were born in the same month. (13 people, 12 months.)

Example 2. In a group of 50 people, at least $\lceil 50/12 \rceil = 5$ share a birth month. (50 people, 12 months.)

Example 3. Divisibility. Among any 11 integers, two have a difference divisible by 10.

There are 10 possible remainders mod 10 (namely 0–9). With 11 integers and only 10 holes, two integers share a remainder, so their difference is divisible by 10.

Example 4. Hair Count. New York City has 8 million people; a human head has at most 150,000 hairs. Since $8{,}000{,}000 \gg 150{,}001$ possible hair counts, at least two New Yorkers have exactly the same number of hairs.

Key Takeaways:

- De Morgan's: $(A \cup B)^c = A^c \cap B^c$ and $(A \cap B)^c = A^c \cup B^c$. Complement swaps $\cup$ and $\cap$.
- IE (2 sets): $|A \cup B| = |A| + |B| - |A \cap B|$.
- IE (3 sets): add singles, subtract pairwise intersections, add triple intersection.
- Basic Pigeonhole: $n > k$ items in k containers $\Rightarrow$ some container has ≥ 2.
- Generalized Pigeonhole: some container has $\geq \lceil n/k \rceil$ items.
- Pigeonhole proves existence — it does not find the specific example.

Common Mistakes:

- **Double-counting.** Don't forget to subtract $|A \cap B|$ in the union formula.
- **De Morgan's sign.** $(A \cup B)^c = A^c \cap B^c$, not $A^c \cup B^c$. The operation flips.
- **Wrong sign in 3-set IE.** You *add* $|A \cap B \cap C|$ back (not subtract).
- **Off-by-one.** To guarantee 2 in one hole among k holes, you need $k + 1$ items, not k.
- **Wrong ceiling.** $\lceil 50/12 \rceil = 5$, not 4. Ceiling rounds *up*.

Homework

De Morgan's Laws and Inclusion-Exclusion

1. Let $U = \{1, 2, \ldots, 12\}$, $A = \{2, 4, 6, 8, 10, 12\}$, $B = \{3, 6, 9, 12\}$.

   (a) Find $(A \cup B)^c$ and $A^c \cap B^c$. Are they equal?

   (b) Find $(A \cap B)^c$ and $A^c \cup B^c$. Are they equal?

2. Let A be the set of multiples of 4 less than 30, and B be the set of multiples of 6 less than 30.

   (a) List the elements of A and B.

   (b) Find $A \cap B$.

   (c) Find $|A \cup B|$ using inclusion-exclusion.

3. In a group of 100 students: 40 play soccer, 50 play basketball, 30 play baseball, 20 play soccer and basketball, 15 play soccer and baseball, 10 play basketball and baseball, and 5 play all three.

   (a) How many play at least one sport?

   (b) How many play exactly two sports?

   (c) How many play exactly one sport?

4. Prove De Morgan's second law $(A \cap B)^c = A^c \cup B^c$ by showing both inclusions.
   Hint: $x \in (A \cap B)^c$ means $x \notin A \cap B$, which means $x \notin A$ or $x \notin B$.

Pigeonhole Principle

5. A bag contains red, blue, green, yellow, and orange candies. How many must you pick to guarantee:

   (a) Two of the same color?

   (b) Four of the same color?

6. In a group of 50 people, at least how many must have been born:

   (a) In the same month?

   (b) On the same day of the week?

7. Among any 11 integers, prove that at least two have a difference divisible by 10.

8. In a standard deck of 52 cards, how many cards must you draw to guarantee:

   (a) Two cards of the same suit?

   (b) Two cards of the same rank (e.g., two Kings)?

   (c) All four cards of some rank?

9. H Prove that in any group of 6 people, either 3 of them all know each other, or 3 of them are all strangers.
   Hint: Pick one person and consider their relationships with the other 5.

10. M Five points are placed inside a square of side length 2. Prove that at least two points are within distance $\sqrt{2}$ of each other.
    Hint: Divide the square into 4 smaller squares.

11. H **Lattice midpoints.**

(a) Five points are chosen in the plane, all with integer coordinates (these are called *lattice points*). Prove that the midpoint of some pair of them is also a lattice point.
Hint: sort each point by the parities of its x- and y-coordinates.

(b) Show that the bound of 5 is tight: find *four* lattice points such that no pair has an integer-coordinate midpoint.

12. H Among any 52 integers, prove you can find two whose sum or difference is divisible by 100.
Hint: Consider remainders mod 100, and pair up remainders that sum to 100.