

MATH 5: HANDOUT 25

SET THEORY

More Probability

Rolling Two Dice: At Least One Six

From last time: when you roll two dice, what is the probability that **at least one** result is a 6?

Example 1. Three Ways to Solve It.

Solution I: Complement.

“At least one 6” is the complement of “no 6 at all.” The probability of no 6 on a single die is $\frac{5}{6}$, so by the product rule:

$$P(\text{no 6 on either die}) = \frac{5}{6} \times \frac{5}{6} = \frac{25}{36}.$$

Therefore:

$$P(\text{at least one 6}) = 1 - \frac{25}{36} = \boxed{\frac{11}{36}}.$$

Solution II: Direct Count (the 6×6 table).

List all $6 \times 6 = 36$ equally likely outcomes. Mark every cell where at least one die shows 6 (the entire bottom row and the entire right column):

		Die 1					
		1	2	3	4	5	6
Die 2	6						
	5						
	4						
	3						
	2						
	1						

Counting the shaded cells: there are 11 of them. So:

$$P(\text{at least one 6}) = \frac{11}{36}.$$

Solution III: Inclusion-Exclusion.

Let $A =$ “die 1 shows 6” and $B =$ “die 2 shows 6.” Then:

$$P(A) = \frac{1}{6}, \quad P(B) = \frac{1}{6}, \quad P(A \cap B) = \frac{1}{36}.$$

By the inclusion-exclusion formula:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{6}{36} + \frac{6}{36} - \frac{1}{36} = \frac{11}{36}.$$

All three methods give the same answer: $\frac{11}{36}$.

The Birthday Problem

Example 2. Problem. In a room of n people, what is the probability that at least two of them share the same birthday?

Solution (using complement).

It is easier to calculate the probability that *all* birthdays are different, then subtract from 1.

Imagine the people enter one by one. Assume 365 equally likely birthdays and ignore leap years.

- Person 1: any birthday. Probability all different so far: $\frac{365}{365} = 1$.

- Person 2: must avoid person 1's birthday. Probability: $\frac{364}{365}$.

- Person 3: must avoid 2 birthdays. Probability: $\frac{363}{365}$.

⋮

- Person n : must avoid $n - 1$ birthdays. Probability: $\frac{365 - (n - 1)}{365}$.

By the product rule:

$$P(\text{all different}) = \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdots \frac{365 - n + 1}{365}.$$

Therefore:

$$P(\text{at least two share a birthday}) = 1 - \frac{365 \cdot 364 \cdot 363 \cdots (365 - n + 1)}{365^n}.$$

Here are some key values:

Number of people n	$P(\text{at least two share a birthday})$
10	$\approx 11.7\%$
20	$\approx 41.1\%$
23	\approx 50.7%
30	$\approx 70.6\%$
40	$\approx 89.1\%$
50	$\approx 97.0\%$
70	$\approx 99.9\%$

Surprising result: With only **23 people**, it is already more likely than not that two of them share a birthday! Most people guess a much larger number — this is why it is called the *Birthday Paradox*.

Why is the answer so small? Because we are not asking whether *someone specific* shares your birthday — we are asking whether *any* two people among the group match. With 23 people there are $\binom{23}{2} = 253$ pairs to check, and that gives plenty of chances for a match.

What is a Set?

Sets are one of the most fundamental ideas in all of mathematics. Nearly every mathematical object — numbers, functions, geometric shapes — can be described using sets. The concept is simple, but its power is enormous.

Definition

A **set** is a collection of distinct objects, called **elements** or **members**. We write sets using curly braces: { and }.

Sets can contain anything: numbers, letters, colors, even other sets.

Example 3. Examples of sets:

- $A = \{1, 2, 3, 4, 5\}$ — the set of integers from 1 to 5
- $B = \{\text{red, blue, green}\}$ — a set of colors
- $C = \{2, 4, 6, 8, \dots\}$ — the set of positive even numbers

To say that something belongs to a set, we write $x \in A$ (read: “ x is in A ”). To say it does not belong, we write $x \notin A$.

Example 4. If $A = \{1, 2, 3, 4, 5\}$, then $3 \in A$ but $7 \notin A$.

Two things make sets special — and different from ordinary lists:

Important properties:

- **No duplicates:** $\{1, 2, 2, 3\} = \{1, 2, 3\}$ (repeating an element doesn’t change the set)
- **Order doesn’t matter:** $\{1, 2, 3\} = \{3, 1, 2\}$

When a set is large or defined by a pattern, listing every element is impractical. Instead, we can describe a set by a rule:

Set-builder notation:

$$\{x : \text{condition on } x\}$$

Read as “the set of all x such that [condition].”

Example 5. • $\{x : x \text{ is an even number less than } 10\} = \{2, 4, 6, 8\}$

- $\{n \in \mathbb{N} : n < 5\} = \{1, 2, 3, 4\}$
- $\{x \in \mathbb{Z} : -2 \leq x \leq 2\} = \{-2, -1, 0, 1, 2\}$

Quick Check

1. List all elements of $A = \{x : x \text{ is a prime number less than } 15\}$.
2. Is $\{1, 2, 3\}$ the same set as $\{3, 2, 1, 2\}$?

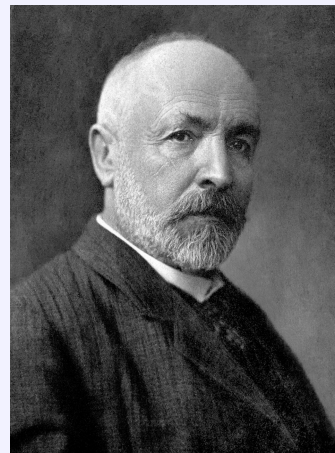
Who Invented Set Theory?

Georg Cantor (1845–1918) was a German mathematician who created modern set theory almost single-handedly. Before Cantor, mathematicians used collections of objects informally, but there was no precise theory.

Cantor’s most surprising discovery was that **infinity comes in different sizes**. The set of natural numbers \mathbb{N} is infinite — but the set of real numbers \mathbb{R} is a *strictly larger* infinity. He introduced the symbol \aleph_0 (“aleph-null”) for the smallest infinite size and proved that $|\mathbb{R}| > \aleph_0$.

His ideas were so radical that many leading mathematicians rejected them. His former teacher Kronecker called his work “a corruption of youth,” and the great Poincaré called set theory a “disease from which mathematics will eventually be cured.” Cantor spent years defending his work and suffered from depression and self-doubt.

Today, set theory is the **foundation of all modern mathematics**. Every mathematical object — numbers, functions, spaces — is built from sets. Cantor’s “disease” turned out to be the bedrock.



Georg Cantor (1845–1918)

Subsets

Once we have sets, a natural question arises: can one set be entirely *contained inside* another? For example, every prime number is an integer — the set of primes “lives inside” the integers. This idea is captured by the notion of a subset.

Definition

Set A is a **subset** of set B , written $A \subseteq B$, if every element of A is also in B .
If $A \subseteq B$ but $A \neq B$, we say A is a **proper subset** of B , written $A \subset B$.

The notation is designed to remind you of inequalities between numbers:

- $A \subseteq B$ is like $a \leq b$: “ A is contained in B , possibly equal.” The bar under \subset signals that equality is allowed.
- $A \subset B$ is like $a < b$: “ A is strictly inside B , not equal.”

Example 6. Let $A = \{1, 2\}$, $B = \{1, 2, 3, 4\}$, $C = \{1, 2\}$.

- $A \subseteq B$ (every element of A is in B)
- $A \subset B$ (proper subset, since $A \neq B$)
- $A \subseteq C$ and $C \subseteq A$, so $A = C$

The third bullet gives us something useful: $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$. This is actually how equality of sets is often *proved* in mathematics — you show containment in both directions.

Two facts about subsets are always true, no matter what the sets are:

Important facts:

- Every set is a subset of itself: $A \subseteq A$
- The empty set is a subset of every set: $\emptyset \subseteq A$ for all A (explained in the next section)

Quick Check

3. Is $\{1, 3\} \subseteq \{1, 2, 3, 4\}$?

4. Is $\{1, 5\} \subseteq \{1, 2, 3, 4\}$?

The Empty Set

Among all sets, one stands apart: the set with no elements at all.

Definition

The **empty set**, written \emptyset (or $\{\}$), is the unique set containing no elements.

The empty set may seem like a trivial technicality, but it appears constantly in mathematics — for instance, $\{x \in \mathbb{Z} : x^2 = -1\} = \emptyset$, because no integer squares to a negative number.

Why is \emptyset a subset of every set? This surprises many people at first. To say $\emptyset \subseteq A$ means: *every element of \emptyset is also in A* . But \emptyset has no elements at all — there is nothing to check. There is no element that could *fail* to be in A , so the statement is true by default.

Mathematicians call this **vacuous truth**: a statement of the form “every x in S satisfies [condition]” is automatically true when S is empty, because there are no counterexamples.

There is also a clean proof by contradiction: suppose $\emptyset \not\subseteq A$. Then there would have to exist some element $x \in \emptyset$ with $x \notin A$. But \emptyset contains no elements, so no such x can exist. Contradiction. Therefore $\emptyset \subseteq A$. \square

\emptyset **versus** $\{\emptyset\}$. The empty set \emptyset and the set $\{\emptyset\}$ are completely different objects:

- \emptyset has **0** elements.
- $\{\emptyset\}$ has **1** element (that element happens to be the empty set itself).

Think of it like bags: an empty bag is not the same as a bag that contains an empty bag.

Special Sets and Cardinality

How do we measure the “size” of a set? For finite sets, the answer is simple — just count the elements.

Definition

The **cardinality** of a set A , written $|A|$, is the number of elements in A .

Example 7. • $|\{1, 2, 3, 4, 5\}| = 5$

- $|\{\text{red}, \text{blue}\}| = 2$
- $|\emptyset| = 0$

The most important sets in mathematics are the *number sets*. You already know them — but have you ever wondered why we need so many? Each one was invented to fix a gap in the previous one:

The number sets — a growing story:

- $\mathbb{N} = \{1, 2, 3, 4, \dots\}$ — **natural numbers**. Great for counting. But what is $3 - 5$? There is no answer inside \mathbb{N} .
- $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ — **integers**. Negative numbers let us subtract freely. But what is $1 \div 3$? There is no answer inside \mathbb{Z} .
- $\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0 \right\}$ — **rational numbers** (fractions). Division is now always possible. But what is $\sqrt{2}$? It cannot be written as a fraction — no answer inside \mathbb{Q} .
- \mathbb{R} — **real numbers**. Adding the irrationals ($\sqrt{2}, \sqrt{3}, \pi, \dots$) fills every point on the number line.

Each set contains all the previous ones: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$. Notice the \subset symbol — each number system is a *proper subset* of the next!

Quick Check

5. What is $|\{a, b, c, d, e, f\}|$?
6. Is $0 \in \mathbb{N}$? Is $-3 \in \mathbb{Z}$?

Set Operations

Just as we can add or multiply numbers, we can combine sets in several ways. These four operations — union, intersection, difference, and complement — are the core toolkit of set theory.

Definition

Union: $A \cup B$ is the set of elements in *A or B* (or both).

Intersection: $A \cap B$ is the set of elements in *both A and B*.

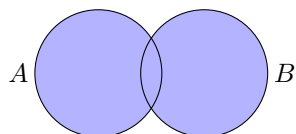
Difference: $A - B$ (or $A \setminus B$) is the set of elements in *A but not in B*.

Complement: If U is the universal set, then $A^c = U - A$ (everything not in A).

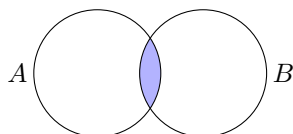
Example 8. Let $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$.

- $A \cup B = \{1, 2, 3, 4, 5, 6\}$
- $A \cap B = \{3, 4\}$
- $A - B = \{1, 2\}$
- $B - A = \{5, 6\}$

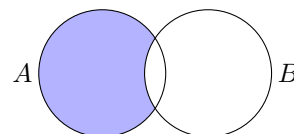
The Venn diagrams below show each operation visually — the shaded region is the result.



$A \cup B$



$A \cap B$



$A - B$

Quick Check

7. If $A = \{1, 2, 3\}$ and $B = \{2, 3, 4, 5\}$, find $A \cup B$ and $A \cap B$.
8. What is $A \cap \emptyset$ for any set A ?

Key Takeaways

- A set is a collection of distinct objects; order and duplicates don't matter.
- Set-builder notation: $\{x : \text{condition}\}$ means "all x satisfying the condition."
- $|A|$ is the cardinality (number of elements) of set A .
- $A \subseteq B$ means every element of A is also in B . Each number set is a proper subset of the next: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.
- $A \cup B =$ elements in A or B ; $A \cap B =$ elements in both; $A - B =$ elements in A but not B ; $A^c =$ everything not in A .

Common Mistakes

- **Confusing \in and \subseteq .** $3 \in \{1, 2, 3\}$ but $\{3\} \subseteq \{1, 2, 3\}$. Elements use \in ; sets use \subseteq .
- **Forgetting \emptyset is a subset of everything.** Yes, $\emptyset \subseteq \{1, 2, 3\}$.
- **Confusing \cup and \cap .** Union (\cup) is "or"; intersection (\cap) is "and."

Classwork

- Let $A = \{1, 2, 3, 4, 5\}$ and $B = \{4, 5, 6, 7\}$. Find:
 - $A \cup B$
 - $A \cap B$
 - $A - B$
 - $B - A$
- Let $P = \{2, 3, 5, 7, 11, 13\}$ (primes less than 15) and $O = \{1, 3, 5, 7, 9, 11, 13\}$ (odd numbers less than 15). Find $P \cap O$ (odd primes less than 15).
- List all subsets of $\{a, b\}$.

Classwork Solutions

- $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$
 - $A \cap B = \{4, 5\}$
 - $A - B = \{1, 2, 3\}$
 - $B - A = \{6, 7\}$
- $P \cap O = \{3, 5, 7, 11, 13\}$ (all odd primes less than 15; note 2 is even)
- The subsets of $\{a, b\}$ are: \emptyset , $\{a\}$, $\{b\}$, $\{a, b\}$.
(A set with n elements has 2^n subsets. Here $n = 2$, so $2^2 = 4$ subsets.)

Homework

- Let $X = \{1, 3, 5, 7, 9\}$ and $Y = \{2, 3, 5, 7\}$. Find:
 - $X \cup Y$
 - $X \cap Y$
 - $X - Y$
 - $|X \cap Y|$
- True or False:
 - $\{1, 2\} \subseteq \{1, 2, 3\}$
 - $\{1, 2, 3\} \subseteq \{1, 2\}$
 - $\emptyset \subseteq \{1, 2, 3\}$
 - $\{1\} \in \{1, 2, 3\}$
- List all subsets of $\{1, 2, 3\}$. How many are there?
- Let A be the set of multiples of 3 less than 30, and B be the set of multiples of 5 less than 30.
 - List the elements of A and B .
 - Find $A \cap B$ (multiples of both 3 and 5).
- A set with n elements has 2^n subsets. Explain why this is true.
Hint: For each element, you have 2 choices: include it or not.
- M** Prove that for any sets A and B : $|A - B| = |A| - |A \cap B|$.
- For each number below, state the *smallest* set among \mathbb{N} , \mathbb{Z} , \mathbb{Q} , \mathbb{R} that contains it:
 - -7
 - $\frac{3}{4}$
 - $\sqrt{9}$
 - $\sqrt{2}$
 - 0
 - π
- The **symmetric difference** of two sets is defined as

$$A \triangle B = (A - B) \cup (B - A),$$

the set of elements belonging to exactly one of A or B .

- If $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$, find $A \triangle B$.
 - Show that $A \triangle A = \emptyset$ for any set A .
 - Show that $A \triangle \emptyset = A$ for any set A .
- If $A \subseteq B$ and $B \subseteq C$, prove that $A \subseteq C$.
 - M** The **power set** of A , written $\mathcal{P}(A)$, is the set of *all* subsets of A .
 - List all elements of $\mathcal{P}(\{1, 2, 3\})$.
 - How many elements does $\mathcal{P}(\{1, 2, 3, 4\})$ have?
 - What is $\mathcal{P}(\emptyset)$? How many elements does it have?

Quick Check Answers

1. $A = \{2, 3, 5, 7, 11, 13\}$
2. Yes, they are the same set.
3. Yes
4. No (5 is not in the second set)
5. $|\{a, b, c, d, e, f\}| = 6$
6. No, $0 \notin \mathbb{N}$ (by our definition). Yes, $-3 \in \mathbb{Z}$.
7. $A \cup B = \{1, 2, 3, 4, 5\}$, $A \cap B = \{2, 3\}$
8. $A \cap \emptyset = \emptyset$