MATH 5: HANDOUT 25

SET THEORY (SUMMARY)

More Probability: The Birthday Problem

In a room of n people, the probability that **at least two share a birthday** is found using the complement:

$$P(\text{at least two match}) = 1 - \frac{365 \cdot 364 \cdots (365 - n + 1)}{365^n}.$$

n	10	20	**23**	30	40	50	70
P	11.7%	41.1%	**50.7%**	70.6%	89.1%	97.0%	99.9%

With only **23 people** the chance already exceeds 50% — the *Birthday Paradox*. With 23 people there are $\binom{23}{2} = 253$ pairs, giving many chances for a match.

Sets and Notation

Definition 1. A **set** is a collection of distinct objects called **elements**. We write $x \in A$ (x is in A) or $x \notin A$. Order and repetition don't matter: $\{1, 2, 3\} = \{3, 1, 2\}$ and $\{1, 2, 2, 3\} = \{1, 2, 3\}$.

Set-builder notation: $\{x : \text{condition on } x\}$ — "the set of all x such that [condition]." Example: $\{x : x \text{ is even and } x < 10\} = \{2, 4, 6, 8\}$.

The Empty Set

Definition 2. The **empty set** $\varnothing$ (also written $\{\}$) is the unique set with no elements.

$\varnothing \subseteq A$ for all A — vacuous truth (no element can fail the condition). Note: $\varnothing \neq \{\varnothing\}$; the latter has one element.

Subsets

Definition 3. $A \subseteq B$ (**subset**) if every element of A is also in B. If also $A \neq B$, write $A \subset B$ (**proper subset**).

$A \subseteq B$ is like $a \leq b$ (equality allowed); $A \subset B$ is like $a < b$ (strictly inside). The bar under $\subset$ signals equality is allowed.

$A = B$ iff $A \subseteq B$ and $B \subseteq A$. Always: $\varnothing \subseteq A$ and $A \subseteq A$.

Cardinality and Number Sets

Definition 4. The **cardinality** $|A|$ is the number of elements in A. Thus $|\varnothing| = 0$.

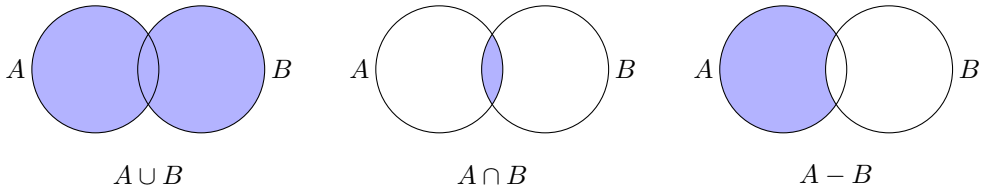
Key number sets (each a proper subset of the next: $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$):

- $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$ **natural numbers**
- $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ **integers**
- $\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z},\ q \neq 0 \right\}$ **rational numbers**
- $\mathbb{R}$ **real numbers** (rationals + irrationals such as $\sqrt{2}$, π)

Set Operations

Definition 5.

- **Union:** $A \cup B$ — elements in A or B (or both)
- **Intersection:** $A \cap B$ — elements in both A and B
- **Difference:** $A - B$ — elements in A but not in B
- **Complement:** $A^c = U - A$ — everything in universal set U not in A



Key Takeaways:

- A set is a collection of distinct objects; order and duplicates don't matter.
- Set-builder notation: $\{x : \text{condition}\}$.
- $\varnothing$ is the unique set with no elements; $\varnothing \subseteq A$ for all A (vacuous truth). Note $\varnothing \neq \{\varnothing\}$.
- $|A|$ = cardinality (element count). $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$.
- $A \cup B$ = or; $A \cap B$ = and; $A - B$ = in A not B; A^c = not in A.
- $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

Common Mistakes:

- **Confusing $\in$ and $\subseteq$.** $3 \in \{1, 2, 3\}$ but $\{3\} \subseteq \{1, 2, 3\}$.
- **Thinking $\varnothing \not\subseteq A$.** The empty set IS a subset of everything.
- **Confusing $\varnothing$ and $\{\varnothing\}$.** One has 0 elements, the other has 1.
- **Confusing $\cup$ and $\cap$.** Union is "or"; intersection is "and."

Homework

1. Let $X = \{1, 3, 5, 7, 9\}$ and $Y = \{2, 3, 5, 7\}$. Find:
   (a) $X \cup Y$
   (b) $X \cap Y$
   (c) $X - Y$
   (d) $|X \cap Y|$
2. True or False:
   (a) $\{1, 2\} \subseteq \{1, 2, 3\}$
   (b) $\{1, 2, 3\} \subseteq \{1, 2\}$
   (c) $\varnothing \subseteq \{1, 2, 3\}$
   (d) $\{1\} \in \{1, 2, 3\}$
3. List all subsets of $\{1, 2, 3\}$. How many are there?
4. Let A be the set of multiples of 3 less than 30, and B be the set of multiples of 5 less than 30.
   (a) List the elements of A and B.
   (b) Find $A \cap B$ (multiples of both 3 and 5).
5. A set with n elements has 2^n subsets. Explain why this is true.
   Hint: For each element, you have 2 choices: include it or not.
6. M Prove that for any sets A and B: $|A - B| = |A| - |A \cap B|$.
7. For each number below, state the *smallest* set among $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ that contains it:
   (a) -7 (c) $\sqrt{9}$ (e) 0
   (b) $\frac{3}{4}$ (d) $\sqrt{2}$ (f) π
8. The **symmetric difference** of two sets is defined as
   $$A \triangle B = (A - B) \cup (B - A),$$
   the set of elements belonging to exactly one of A or B.
   (a) If $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5, 6\}$, find $A \triangle B$.
   (b) Show that $A \triangle A = \varnothing$ for any set A.
   (c) Show that $A \triangle \varnothing = A$ for any set A.
9. If $A \subseteq B$ and $B \subseteq C$, prove that $A \subseteq C$.
10. M The **power set** of A, written $\mathcal{P}(A)$, is the set of *all* subsets of A.
    (a) List all elements of $\mathcal{P}(\{1, 2, 3\})$.
    (b) How many elements does $\mathcal{P}(\{1, 2, 3, 4\})$ have?
    (c) What is $\mathcal{P}(\varnothing)$? How many elements does it have?