MATH 5: HANDOUT 24

PROBABILITY IV: COUNTING, INCLUSION-EXCLUSION, AND MORE

Arrangements and Probability

Counting techniques are essential for calculating probabilities.

Example 1. A theater has 50 seats. All 50 are occupied, but people sat randomly. What is the probability that everyone is in their assigned seat?

There is exactly 1 correct arrangement out of $50!$ possible arrangements.

$$P(\text{all correct}) = \frac{1}{50!}$$

This is an incredibly small number!

Example 2. A puzzle has 9 pieces that form a 3×3 square. If you try random arrangements (one per second), how long to try them all?

Number of arrangements: $9! = 362{,}880$

Time: $362{,}880$ seconds ≈ 4.2 days

Example 3. A 4-digit PIN is created by randomly selecting digits without repetition. What is the probability that the PIN is 1234?

Step 1: Count total outcomes. Without repetition, there are $P(10, 4) = 10 \times 9 \times 8 \times 7 = 5{,}040$ possible PINs.

Step 2: Count favorable outcomes. Only 1 PIN is exactly "1234."

Step 3: Calculate probability.

$$P(\text{PIN is 1234}) = \frac{1}{5{,}040}$$

This pattern—$P = \frac{\text{favorable outcomes}}{\text{total outcomes}}$—is why counting is so important in probability!

Example 4. A 4-digit PIN is created by randomly selecting digits (repetition allowed). What is the probability that all four digits are different?

Step 1: Count total outcomes. With repetition allowed, there are $10^4 = 10{,}000$ possible PINs.

Step 2: Count favorable outcomes. PINs with all different digits: $P(10, 4) = 10 \times 9 \times 8 \times 7 = 5{,}040$.

Step 3: Calculate probability.

$$P(\text{all different}) = \frac{5{,}040}{10{,}000} = \frac{504}{1000} = 50.4\%$$

So about half of all 4-digit PINs have no repeated digits!

Counting and Password Security

How long would it take a computer to guess your password by trying every possibility?
Assume a computer can try 1 million passwords per second.

Password type	Possibilities	Time to crack
4-digit PIN	$10^4 = 10{,}000$	0.01 seconds
4 lowercase letters	$26^4 \approx 457{,}000$	0.5 seconds
8 lowercase letters	$26^8 \approx 209$ billion	2.4 days
8 mixed (a-z, A-Z, 0-9)	$62^8 \approx 218$ trillion	6.9 years

This is why longer passwords with more character types are much more secure!

Example 5. Five friends (including Alice and Bob) sit randomly in a row. What is the probability that Alice and Bob end up sitting next to each other?

Step 1: Count total outcomes. All arrangements of 5 people: $5! = 120$.

Step 2: Count favorable outcomes. From Handout 23 (Counting with Restrictions), arrangements with Alice and Bob adjacent: $4! \times 2 = 48$.

Step 3: Calculate probability.

$$P(\text{Alice and Bob adjacent}) = \frac{48}{120} = \frac{2}{5} = 40\%$$

Example 6. A shelf has 5 math books (identical) and 3 science books (identical). If they are arranged randomly, what is the probability that all math books are on the left and all science books are on the right?

Step 1: Count total outcomes. Total arrangements of 5 identical + 3 identical items: $\frac{8!}{5! \cdot 3!} = \frac{40320}{120 \cdot 6} = 56$.

Step 2: Count favorable outcomes. Only 1 arrangement has all math books left, all science books right: MMMMMSSS.

Step 3: Calculate probability.

$$P = \frac{1}{56}$$

Example 7. Six people sit randomly around a circular table. What is the probability that two specific people (say, Alice and Bob) sit directly across from each other?

Step 1: Count total outcomes. Circular arrangements of 6 people: $(6-1)! = 5! = 120$.

Step 2: Count favorable outcomes. Fix Alice's position (to account for rotations). Bob must sit directly across (1 choice). The remaining 4 people can sit anywhere: $4! = 24$ ways.

Step 3: Calculate probability.

$$P = \frac{24}{120} = \frac{1}{5} = 20\%$$

Quick Check

1. If 4 people sit randomly in 4 chairs, what is the probability they're all in a specific arrangement?
2. A combination lock has 3 digits (0–9), no repeats allowed. How many combinations are possible? If you guess randomly, what is the probability you get it right on the first try?

Teaser: The Birthday Problem

Here's one of the most surprising results in probability:
How many people do you need in a room for there to be a 50% chance that at least two share a birthday?
Most people guess a large number—maybe 180 or so (half of 365). The actual answer is just **23 people**!
This problem uses everything we've learned: counting with permutations, the complement rule from Handout 22, and careful thinking about "at least one." Try it in the homework!

Inclusion-Exclusion Principle

When two events can overlap, we need to be careful not to count outcomes twice.

Theorem

Inclusion-Exclusion Principle.
For any two events A and B:

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

We subtract $P(A \text{ and } B)$ because those outcomes were counted twice.

Example 8. In a class of 100 students:

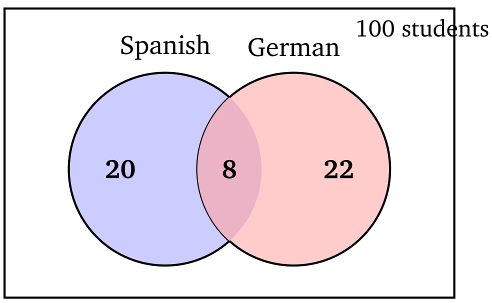
- 28 speak Spanish
- 30 speak German
- 8 speak both Spanish and German

How many speak Spanish or German (or both)?

Using inclusion-exclusion:

$$\text{Spanish or German} = 28 + 30 - 8 = 50$$

A **Venn diagram** helps visualize overlapping events:



Spanish only: $28 - 8 = 20$
German only: $30 - 8 = 22$
Both: 8
Total: $20 + 8 + 22 = 50$

Neither: $100 - 50 = 50$

For three events, the formula becomes:

$$P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C) - P(A \text{ and } B) - P(A \text{ and } C) - P(B \text{ and } C) + P(A \text{ and } B \text{ and } C)$$

Example 9. In a class of 50 students: 20 play soccer, 15 play basketball, 12 play tennis. Also: 6 play soccer and basketball, 4 play soccer and tennis, 3 play basketball and tennis, and 2 play all three sports. How many play at least one sport?

Using inclusion-exclusion:

$$20 + 15 + 12 - 6 - 4 - 3 + 2 = 36 \text{ students}$$

Why $+2$ at the end? Students who play all three were subtracted three times (once for each pair) but only added three times (once for each sport), so we need to add them back once.

Quick Check

3. In a group of 40 students, 25 play soccer, 18 play basketball, and 10 play both. How many play at least one of these sports?
4. A card is drawn from a deck. What is the probability it is a heart or a face card?

More Probability Problems

Example 10. Dice sums.
When rolling 2 dice, what is the probability that the sum is at most 7?
List favorable outcomes systematically:

- Sum = 2: (1,1) — 1 way
- Sum = 3: (1,2), (2,1) — 2 ways
- Sum = 4: (1,3), (2,2), (3,1) — 3 ways
- Sum = 5: 4 ways
- Sum = 6: 5 ways
- Sum = 7: 6 ways

Total favorable: $1 + 2 + 3 + 4 + 5 + 6 = 21$
$P(\text{sum} \leq 7) = \frac{21}{36} = \frac{7}{12}$

Example 11. All different.
When rolling 3 dice, what is the probability that all three show different numbers?

- First die: any of 6 numbers
- Second die: 5 numbers (not the first)
- Third die: 4 numbers (not the first two)

$P(\text{all different}) = \frac{6\times5\times4}{6^3} = \frac{120}{216} = \frac{5}{9}$

Quick Check

5. When rolling 2 dice, how many ways can you get a sum of 8?
6. Two numbers are chosen from {1, 2, 3} with replacement. What is the probability both are the same?

Fair Games

Probability lets us decide whether a game is *fair*—meaning neither player has a long-term advantage.

The key idea: If you play a game N times, you expect to win roughly $P(\text{win}) \times N$ times and lose roughly $P(\text{lose}) \times N$ times. A game is fair when the total amount you expect to collect equals the total amount you expect to pay.

Example 12. You roll one die. If it shows 1 or 2, you win \$4. Otherwise, you pay \$2. Is this game fair?
$P(\text{win}) = \dfrac{2}{6} = \dfrac{1}{3}, \quad P(\text{lose}) = \dfrac{4}{6} = \dfrac{2}{3}.$
Imagine playing **3 games**. You'd expect to win 1 time and lose 2 times.

- Collected: $1 \times \$4 = \$4$
- Paid out: $2 \times \$2 = \$4$

Equal — the game is **fair**!

Example 13. Same game, but winning now pays only \$3. Is it still fair?

Over 3 games: win 1 time (\$3 collected), lose 2 times (\$4 paid). Net: $-\$1$. You'd slowly lose money — the game is **not fair**, and favors your opponent.

How to find the fair prize. Over 3 games you win once and lose twice. For the totals to balance:

$$1 \times \text{prize} = 2 \times \$2 \quad \Rightarrow \quad \text{prize} = \$4.$$

Winning must pay \$4 for this game to be fair — which matches the first example!

Example 14. A bag has 3 red and 1 blue marble. You draw one at random, then put it back. If it's red, you pay \$1; if it's blue, you win \$3. Is this fair?

$P(\text{red}) = \frac{3}{4}, \quad P(\text{blue}) = \frac{1}{4}.$

Imagine **4 games**. Expect 3 red and 1 blue.

- Collected: $1 \times \$3 = \$3$
- Paid out: $3 \times \$1 = \$3$

Equal — **fair**!

In general: the prize for a $\frac{1}{4}$ chance event should be 3 times the penalty for a $\frac{3}{4}$ chance event to keep the game balanced. The rarer the win, the bigger the prize must be to compensate.

Example 15. Two-dice sum game. Roll two dice. You win \$3 if the sum is 7 or 8; you pay \$3 if the sum is 2, 3, 11, or 12. Is this fair?

Step 1 — Count outcomes.

- Sum 7: (1,6),(2,5),(3,4),(4,3),(5,2),(6,1) = 6 ways
- Sum 8: (2,6),(3,5),(4,4),(5,3),(6,2) = 5 ways. Winning total: $6 + 5 = \mathbf{11}$ ways.
- Sums 2, 3, 11, 12: $1 + 2 + 2 + 1 = \mathbf{6}$ ways.

Step 2 — Check balance. Imagine 36 games: win 11 times ($11 \times \$3 = \$33$) and lose 6 times ($6 \times \$3 = \$18$). Net $+\$15$ — **not fair**; the player who wins on 7 or 8 has a big advantage.

Step 3 — Find fair alternatives.

- **Option A (equal groups):** Change the winning condition to sum $= 7$ only (6 ways) — now both sides have 6 ways, so equal prizes are fair.
- **Option B (keep original win condition):** Keep winning on 7 or 8 (11 ways) vs. losing on 2,3,11,12 (6 ways). For balance: $11 \times$ prize $= 6 \times$ penalty. Example: prize $= \$6$, penalty $= \$11$ gives $11 \times \$6 = 6 \times \$11 = \$66$.

Quick Check

7. You flip a coin. Heads you win \$3, tails you pay \$1. Would you rather flip or not flip? Explain.
8. A spinner has 4 equal sections: three are red and one is blue. If it lands on blue, you win \$X; otherwise you lose \$2. What should X be to make the game fair?

Key Takeaways

- Use counting (permutations, factorials) to find the number of favorable and total outcomes for probability calculations.
- Inclusion-Exclusion: $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$.
- Complement rule (from Handout 21): $P(\text{not } A) = 1 - P(A)$. Especially useful for "at least one" problems.
- When counting dice outcomes, list them systematically to avoid missing any.
- For "all different" problems, think about choosing without repetition.
- **Fair game:** imagine playing N games. Collect (wins) $\times$ (prize) and pay (losses) $\times$ (penalty). If these are equal, the game is fair.

Common Mistakes

- **Forgetting to subtract the overlap.** When events can happen together, use inclusion-exclusion.
- **Counting dice rolls incorrectly.** Remember: (1,2) and (2,1) are different outcomes for two dice.
- **Judging a game by probability alone.** A game with $P(\text{win}) = \frac{2}{3}$ can still be unfair if the prize is too small. Always check: wins $\times$ prize vs. losses $\times$ penalty.

Classwork

1. In a class of 50 students, 30 like math, 25 like science, and 15 like both. How many students like at least one of these subjects?
2. A card is drawn from a standard deck. Find the probability that it is:
   (a) A spade or an ace
   (b) A red card or a king
   (c) A face card or a diamond
3. You roll two dice. Find the probability that:
   (a) The sum is exactly 7
   (b) The sum is at least 10
   (c) Both dice show the same number
4. Two numbers are randomly chosen from $\{1, 2, 3, 4\}$, one after another (with replacement allowed).
   (a) What is the probability both numbers are the same?
   (b) What is the probability the numbers are in strictly increasing order?

Classwork Solutions

1. By inclusion-exclusion: $30 + 25 - 15 = 40$ students like at least one subject.
2. (a) Spades: 13. Aces: 4. Ace of spades: 1.
   $P = \frac{13+4-1}{52} = \frac{16}{52} = \frac{4}{13}$
   (b) Red cards: 26. Kings: 4. Red kings: 2.
   $P = \frac{26+4-2}{52} = \frac{28}{52} = \frac{7}{13}$
   (c) Face cards: 12. Diamonds: 13. Diamond face cards: 3.
   $P = \frac{12+13-3}{52} = \frac{22}{52} = \frac{11}{26}$
3. (a) Sum = 7: (1,6), (2,5), (3,4), (4,3), (5,2), (6,1) = 6 ways.
   $P = \frac{6}{36} = \frac{1}{6}$
   (b) Sum $\geq$ 10: Sum 10 (3 ways), 11 (2 ways), 12 (1 way) = 6 ways.
   $P = \frac{6}{36} = \frac{1}{6}$
   (c) Doubles: (1,1), (2,2), ..., (6,6) = 6 ways.
   $P = \frac{6}{36} = \frac{1}{6}$
4. Total outcomes: $4 \times 4 = 16$
   (a) Same: (1,1), (2,2), (3,3), (4,4) = 4 outcomes.
   $P = \frac{4}{16} = \frac{1}{4}$
   (b) Strictly increasing: (1,2), (1,3), (1,4), (2,3), (2,4), (3,4) = 6 outcomes.
   $P = \frac{6}{16} = \frac{3}{8}$

Homework

1. In a group of 100 students: 28 speak Spanish, 30 speak German, 42 speak French. Also, 8 speak Spanish and German, 10 speak Spanish and French, 5 speak German and French, and 3 speak all three languages.
   (a) Draw a Venn diagram showing this information.
   (b) How many students speak at least one of the three languages?
   (c) How many students don't speak any of these languages?

2. Suppose you have a standard die.
   (a) What is the probability that when you roll once, the number is less than 5?
   (b) What is the probability that when you roll once, the number is less than 7?
   (c) What is the probability that when you roll twice, at least one result is a 6?
   (d) What is the probability that when you roll three times, all results are odd?

3. You roll 2 dice.
   (a) What is the probability that the sum is at most 7?
   (b) You and a friend play a game: you win if the sum is at most 7, otherwise your friend wins. The loser pays $1. Would you rather be the one who wins on "at most 7" or "more than 7"? Explain.
   (c) How would you adjust the payments to make this game fair?

4. M You roll 3 dice.
   (a) What is the probability that all three numbers are different?
   (b) You and a friend play: if all numbers are different, you win and get $2. Otherwise, you lose and pay $3. Would you rather be you or your friend?

5. M Roll two dice. You win $7 if the sum is **prime**, and pay $5 if the sum is **composite** (not prime and not 1).
   (a) Which sums (2–12) are prime? Which are composite? How many of the 36 outcomes give a prime sum? A composite sum?
   (b) Is this game fair?
   (c) What if both prize and penalty were $6? Who would have the advantage?

6. H **The Two-Child Problem.** A family has two children. You are told that at least one of them is a boy. What is the probability that both children are boys?
   Hint: List all possible combinations of two children (BB, BG, GB, GG) and eliminate the impossible ones.
   Note: This problem is famous for being tricky! Think carefully about what information you're given.

7. M A class of 30 students sits randomly in 30 assigned seats. What is the probability that three specific students—Alice, Bob, and Carol—all end up in their correct seats?

8. M **The Birthday Problem.** In a room of 23 people, what is the probability that at least two people share the same birthday? (Assume 365 days in a year, and all birthdays are equally likely.)
   Hint: It's easier to find $P(\text{all different birthdays})$ and use the complement rule. Use permutations!

Quick Check Answers

1. $P = \frac{1}{4!} = \frac{1}{24}$
2. $P(10, 3) = 10 \times 9 \times 8 = 720$ combinations; $P(\text{correct on first try}) = \dfrac{1}{720}$
3. $25 + 18 - 10 = 33$ students
4. Hearts: 13. Face cards: 12. Heart face cards: 3.
   $P = \frac{13+12-3}{52} = \frac{22}{52} = \frac{11}{26}$
5. Sum = 8: (2,6), (3,5), (4,4), (5,3), (6,2) = 5 ways
6. Same number: (1,1), (2,2), (3,3) = 3 out of 9.
   $P = \frac{3}{9} = \frac{1}{3}$
7. Over 2 games expect 1 win ($3 collected) and 1 loss ($1 paid). Net $+\$2$ — you'd rather flip!
8. Over 4 games expect 1 win and 3 losses. For balance: $X = 3 \times \$2 = \$6$.