MATH 5: HANDOUT 24

Probability IV: Counting, Inclusion-Exclusion, and More (Summary)

Counting and Probability

Counting techniques let us compute probabilities: $P = \dfrac{\text{favorable outcomes}}{\text{total outcomes}}$.

Example: 50 people sit randomly. $P($all in assigned seats)?

$$P = \frac{1}{50!}$$

Example: 4-digit PIN (repeats allowed). $P($all digits different)?

$$P = \frac{P(10,4)}{10^4} = \frac{5{,}040}{10{,}000} = 50.4\%$$

Example: 4-digit PIN (no repeats). $P($PIN is 1234)?

$$P = \frac{1}{P(10,4)} = \frac{1}{5{,}040}$$

Example: 5 friends in a row. $P($Alice & Bob adjacent)?

$$P = \frac{4! \times 2}{5!} = \frac{48}{120} = \frac{2}{5}$$

Inclusion-Exclusion Principle

When two events overlap, outcomes in both would be counted twice without the correction.

Theorem 1 (Inclusion-Exclusion (two events))**.**

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Example: 100 students: 28 speak Spanish, 30 speak German, 8 speak both. Spanish or German: $28+30-8 = 50$.

For **three events**:

$$P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C) - P(A \text{ and } B) - P(A \text{ and } C) - P(B \text{ and } C) + P(A \text{ and } B \text{ and } C)$$

Example: 50 students: 20 soccer, 15 basketball, 12 tennis; 6 soccer+basketball, 4 soccer+tennis, 3 basketball+tennis, 2 all three.

$$20 + 15 + 12 - 6 - 4 - 3 + 2 = 36 \text{ play at least one sport.}$$

Systematic Dice Counting

Always list outcomes in order to avoid missing any.

Example: 2 dice, $P(\text{sum} \le 7)$?
Sums 2–7 give $1+2+3+4+5+6 = 21$ outcomes.

$$P = \frac{21}{36} = \frac{7}{12}$$

Example: 3 dice, $P(\text{all different})$?
Choose without repetition:

$$P = \frac{6 \times 5 \times 4}{6^3} = \frac{120}{216} = \frac{5}{9}$$

Fair Games

Definition 1 (Fair Game)**.** A game is **fair** if neither player has a long-term advantage. To check: imagine playing N times (choose N so the counts are whole numbers). The game is fair when:

$$\text{wins} \times \text{prize} = \text{losses} \times \text{penalty}$$

Example: Roll one die. If 1 or 2 ($P = \frac{1}{3}$), win \$4; otherwise pay \$2. Over 3 games: 1 win (\$4 collected), 2 losses (\$4 paid) — **fair**!
Example (finding the fair prize): Same game, but winning pays \$3. Over 3 games: win once (\$3), lose twice (\$4) — down \$1, **not fair**. For balance: $1 \times \text{prize} = 2 \times \$2$, so the fair prize is \$4.

Common Mistakes

- **Forgetting to subtract the overlap.** If events can happen together, apply inclusion-exclusion.
- **Treating $(1, 2)$ and $(2, 1)$ as the same outcome.** For ordered pairs (two dice), they are different.
- **Judging fairness by probability alone.** A win probability of $\frac{2}{3}$ does not guarantee a fair game — the prize size matters too.
- **Forgetting the complement rule.** For "at least one" problems, $P(\text{at least one}) = 1 - P(\text{none})$ is often easier.

Homework

1. In a group of 100 students: 28 speak Spanish, 30 speak German, 42 speak French. Also, 8 speak Spanish and German, 10 speak Spanish and French, 5 speak German and French, and 3 speak all three languages.

   (a) Draw a Venn diagram showing this information.

   (b) How many students speak at least one of the three languages?

   (c) How many students don't speak any of these languages?

2. Suppose you have a standard die.

   (a) What is the probability that when you roll once, the number is less than 5?

   (b) What is the probability that when you roll once, the number is less than 7?

   (c) What is the probability that when you roll twice, at least one result is a 6?

   (d) What is the probability that when you roll three times, all results are odd?

3. You roll 2 dice.

   (a) What is the probability that the sum is at most 7?

   (b) You and a friend play a game: you win if the sum is at most 7, otherwise your friend wins. The loser pays $1. Would you rather be the one who wins on "at most 7" or "more than 7"? Explain.

   (c) How would you adjust the payments to make this game fair?

4. M You roll 3 dice.

   (a) What is the probability that all three numbers are different?

   (b) You and a friend play: if all numbers are different, you win and get $2. Otherwise, you lose and pay $3. Would you rather be you or your friend?

5. M Roll two dice. You win $7 if the sum is **prime**, and pay $5 if the sum is **composite**.

   (a) How many of the 36 outcomes give a prime sum? A composite sum?

   (b) Is this game fair?

   (c) What if both prize and penalty were $6? Who has the advantage?

6. H **The Two-Child Problem.** A family has two children. You are told that at least one of them is a boy. What is the probability that both children are boys?
   Hint: List all possible combinations of two children (BB, BG, GB, GG) and eliminate the impossible ones.
   Note: This problem is famous for being tricky! Think carefully about what information you're given.

7. M A class of 30 students sits randomly in 30 assigned seats. What is the probability that three specific students—Alice, Bob, and Carol—all end up in their correct seats?

8. M **The Birthday Problem.** In a room of 23 people, what is the probability that at least two people share the same birthday? (Assume 365 days in a year, and all birthdays are equally likely.)
   Hint: It's easier to find P(all different birthdays) and use the complement rule. Use permutations!