

# MATH 5: HANDOUT 23

## PROBABILITY III: COUNTING AND PERMUTATIONS

### Counting with Repetitions Allowed

Many counting problems can be solved by thinking about choices made in sequence.

#### Theorem

##### Multiplication Principle for Counting.

If you make a sequence of choices where:

- The first choice has  $n_1$  options
- The second choice has  $n_2$  options
- And so on...

Then the total number of ways to make all choices is:

$$n_1 \times n_2 \times n_3 \times \cdots$$

**Example 1.** How many 3-letter “words” can be formed using the 26 letters of the alphabet, if letters can be repeated?

- First letter: 26 choices
- Second letter: 26 choices (repetition allowed)
- Third letter: 26 choices

Total:  $26 \times 26 \times 26 = 26^3 = 17,576$

A PIN code has 4 digits (0–9). How many different PIN codes are possible?

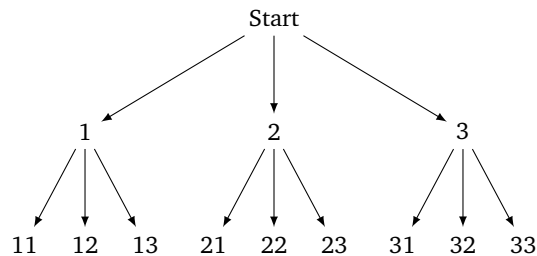
Each digit has 10 choices:  $10 \times 10 \times 10 \times 10 = 10^4 = 10,000$

In general, when choosing  $k$  items from  $n$  options **with repetition allowed** and **order matters**:

$$\text{Number of ways} = n^k$$

### Visualizing with a Tree Diagram

Why do we multiply? A tree diagram makes this clear. Here are all 2-digit codes using only digits 1, 2, 3:



Each first choice branches into 3 options, and each of those branches into 3 more:  $3 \times 3 = 9$  total outcomes.

### Quick Check

1. How many 2-letter “words” can be made from 26 letters (repetition allowed)?
2. A license plate has 3 letters followed by 3 digits. How many plates are possible?

## Counting Without Repetitions

What if each item can only be used once?

**Example 2.** How many 3-letter “words” can be formed if no letter can be repeated?

- First letter: 26 choices
- Second letter: 25 choices (can’t use the first letter again)
- Third letter: 24 choices

Total:  $26 \times 25 \times 24 = 15,600$

### Definition

When choosing  $k$  items from  $n$  options **without repetition** and **order matters**, the number of ways is called a **permutation**, written:

$$P(n, k) = n \times (n - 1) \times (n - 2) \times \cdots \times (n - k + 1)$$

This product has exactly  $k$  factors, starting from  $n$  and counting down.

**Example 3.** In a club of 30 people, they need to select a President, Vice-President, and Treasurer (all different people). How many ways can this be done?

- President: 30 choices
- Vice-President: 29 choices (can’t be the President)
- Treasurer: 28 choices

Total:  $P(30, 3) = 30 \times 29 \times 28 = 24,360$

### Quick Check

3. In a race with 10 runners, how many ways can the gold, silver, and bronze medals be awarded?
4. How many 4-digit PIN codes have no repeated digits?

## Summary: With vs. Without Repetition

	With Repetition	Without Repetition
<b>Rule</b>	Can reuse items	Each item used at most once
<b>Formula</b>	$n^k$	$P(n, k) = n(n - 1)(n - 2) \cdots$
<b>Example</b>	PIN codes, passwords	Medals, officer positions

## Factorials

When we arrange *all*  $n$  items in order, we get a special case:

### Definition

The **factorial** of  $n$ , written  $n!$  (read “ $n$  factorial”), is:

$$n! = n \times (n - 1) \times (n - 2) \times \cdots \times 2 \times 1$$

By convention,  $0! = 1$ .

### Example 4. Examples of factorials:

$$1! = 1$$

$$2! = 2 \times 1 = 2$$

$$3! = 3 \times 2 \times 1 = 6$$

$$4! = 4 \times 3 \times 2 \times 1 = 24$$

$$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$$

$$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$$

Factorials grow incredibly fast—much faster than exponentials!

$n$	$n!$	Approximately
5	120	one hundred
10	3,628,800	3.6 million
15	1,307,674,368,000	1.3 trillion
20	2,432,902,008,176,640,000	2.4 quintillion
52	$\approx 8 \times 10^{67}$	$8 \times 10^{67}$

The number  $52!$  (ways to shuffle a deck of cards) is so large that every time you shuffle a deck, you’re almost certainly creating an arrangement that has never existed before in human history!

**Useful formula:** We can write permutations using factorials:

$$P(n, k) = \frac{n!}{(n - k)!}$$

**Example 5.**  $P(6, 4) = 6 \times 5 \times 4 \times 3 = \frac{6!}{2!} = \frac{720}{2} = 360$

**Notation note:** Different textbooks write permutations differently. You may see:

$$P(n, k) = {}_n P_k = P_n^k = P_{n,k}$$

These all mean the same thing: the number of ways to arrange  $k$  items chosen from  $n$  options, without repetition.

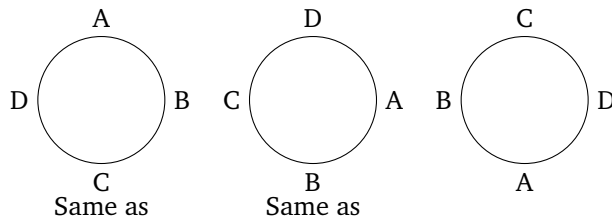
## Circular Arrangements

What if people sit around a circular table instead of in a row?

**Example 6.** In how many ways can 4 people sit around a circular table?

**Wrong approach:**  $4! = 24$ ?

The problem is that rotations of the same arrangement look identical at a round table. For example, if A, B, C, D sit clockwise starting from the top, rotating everyone one seat clockwise gives the same "circular arrangement"—everyone still has the same neighbors!



**Solution:** Fix one person's position to break the symmetry. Say A sits at the "top." Then arrange the remaining 3 people in the other seats:  $3! = 6$  ways.

### Theorem

#### Circular Arrangements.

The number of ways to arrange  $n$  distinct objects in a circle is:

$$(n - 1)!$$

We fix one object's position (1 way) and arrange the remaining  $n - 1$  objects ( $n - 1$  ways for the first,  $n - 2$  for the second, etc.).

### Quick Check

5. In how many ways can 6 people sit around a circular table?
6. A necklace has 5 different colored beads. In how many ways can they be arranged?

## Arrangements with Identical Items

What if some items are identical? Let's discover the rule through a simple example.

**Example 7.** How many ways can we arrange the letters in BOO?

**First attempt:** There are 3 letters, so  $3! = 6$  arrangements... right?

Let's list them: BOO, BOO, OBO, OBO, OOB, OOB. Wait—we got repeats!

**The problem:** The two O's are identical. Swapping them doesn't create a new word.

**The trick:** Temporarily pretend the O's are different by coloring them:

B $\color{red}O$  (red O first, then blue O)

Now list all  $3! = 6$  arrangements of these "different" letters:

Colored arrangement	Looks like	Group
B $\color{red}O$	BOO	1
B $\color{blue}O$	BOO	1
$\color{red}O$ B	OBO	2
$\color{blue}O$ B	OBO	2
$\color{red}O$ $\color{blue}O$ B	OOB	3
$\color{blue}O$ $\color{red}O$ B	OOB	3

The 6 colored arrangements group into pairs—each pair looks identical when we remove the colors!  
**Why pairs?** For each arrangement, the 2 O's can be swapped in  $2! = 2$  ways, but both look the same.

**Answer:**  $\frac{3!}{2!} = \frac{6}{2} = 3$  distinct arrangements: BOO, OBO, OOB.

**Example 8.** How many ways can we arrange the letters in MOON?

Color the two O's: M**OO**N

If all letters were different, we'd have  $4! = 24$  arrangements.

But for each visible arrangement (like MONO), there are  $2! = 2$  colored versions:

M**OO**N and M**NO**O both look like MONO

So every group of 2 colored arrangements collapses into 1 visible arrangement:

$$\frac{4!}{2!} = \frac{24}{2} = 12 \text{ distinct arrangements}$$

**Example 9.** What if there are MORE than 2 identical items? Consider AAB.

Color the A's: **AA**B

List all  $3! = 6$  colored arrangements:

Colored arrangement	Looks like	Group
<b>AA</b> B	AAB	1
<b>BA</b> A	AAB	1
<b>ABA</b>	ABA	2
<b>BAA</b>	ABA	2
<b>BAA</b>	BAA	3
<b>BA</b> A	BAA	3

Again,  $\frac{3!}{2!} = 3$  distinct arrangements.

Now consider AABB (2 A's and 2 B's). Color them: **AA****BB**

If all different:  $4! = 24$  arrangements.

But for each visible arrangement, the 2 A's can be swapped ( $2!$  ways) AND the 2 B's can be swapped ( $2!$  ways). That's  $2! \times 2! = 4$  colored versions that look the same!

$$\frac{4!}{2! \times 2!} = \frac{24}{2 \times 2} = 6 \text{ distinct arrangements}$$

We can list them: AABB, ABAB, ABBA, BAAB, BABA, BBAA. Indeed, 6 arrangements!

### Theorem

#### Permutations with Identical Items.

If you have  $n$  items where  $n_1$  are identical of one type,  $n_2$  are identical of another type, etc., the number of distinct arrangements is:

$$\frac{n!}{n_1! \cdot n_2! \cdot n_3! \cdot \dots}$$

**Why?** If all items were different, there would be  $n!$  arrangements. But each visible arrangement is counted multiple times—once for each way to rearrange the identical items among themselves. Dividing by  $n_1! \cdot n_2! \cdot \dots$  removes this overcounting.

**Example 10.** How many ways can we arrange the letters in BANANA?

BANANA has 6 letters: A appears 3 times, N appears 2 times, B appears 1 time.

If we colored all letters differently, we'd have  $6! = 720$  arrangements. But:

- The 3 A's can be rearranged in  $3! = 6$  ways (all looking the same)
- The 2 N's can be rearranged in  $2! = 2$  ways (all looking the same)

$$\frac{6!}{3! \cdot 2! \cdot 1!} = \frac{720}{6 \cdot 2 \cdot 1} = \frac{720}{12} = 60 \text{ arrangements}$$

**Example 11.** How many ways can we arrange the letters in MISSISSIPPI?  
MISSISSIPPI has 11 letters: M (1), I (4), S (4), P (2).

$$\frac{11!}{1! \cdot 4! \cdot 4! \cdot 2!} = \frac{39,916,800}{1 \cdot 24 \cdot 24 \cdot 2} = \frac{39,916,800}{1,152} = 34,650 \text{ arrangements}$$

### Quick Check

7. How many ways can we arrange the letters in TREE?
8. How many ways can we arrange the letters in AABB?

## Counting with Restrictions

Sometimes we need to count arrangements where certain items must (or must not) be together.

**Example 12. Items that must be adjacent.**

In how many ways can 5 people (A, B, C, D, E) stand in a row if A and B must stand next to each other?

**Trick:** Treat A and B as a single "super-person" [AB]. Now we arrange 4 objects: [AB], C, D, E.

$$\text{Arrangements of 4 objects} = 4! = 24$$

But A and B can be in either order within their unit: AB or BA. So we multiply by 2:

$$\text{Total} = 4! \times 2 = 24 \times 2 = 48 \text{ ways}$$

**Example 13. Items that must NOT be adjacent.**

In how many ways can 5 people (A, B, C, D, E) stand in a row if A and B must NOT stand next to each other?

Use the complement: Total arrangements – arrangements where A and B ARE adjacent.

$$5! - 48 = 120 - 48 = 72 \text{ ways}$$

**Example 14. Multiple constraints.**

In how many ways can 6 people stand in a row if A, B, and C must all stand together (in any order)?

Treat A, B, C as a single unit [ABC]. Arrange 4 objects: [ABC], D, E, F.

$$\text{Arrangements of 4 objects} = 4! = 24$$

Within the unit, A, B, C can be in any order:  $3! = 6$  ways.

$$\text{Total} = 4! \times 3! = 24 \times 6 = 144 \text{ ways}$$

### Quick Check

9. In how many ways can 4 people stand in a row if two specific people must be next to each other?
10. In how many ways can 4 people stand in a row if two specific people must NOT be next to each other?

### Key Takeaways

- With repetition allowed:  $n^k$  ways to choose  $k$  items from  $n$  options.
- Without repetition (permutations):  $P(n, k) = n \times (n - 1) \times \cdots \times (n - k + 1)$ .
- Factorial:  $n! = n \times (n - 1) \times \cdots \times 2 \times 1$ .
- To arrange all  $n$  items:  $n!$  ways.
- Permutation formula:  $P(n, k) = \frac{n!}{(n-k)!}$ .

### Common Mistakes

- **Using  $n^k$  when repetition is not allowed.** If items can't repeat, use permutations instead.
- **Forgetting whether order matters.** Choosing President then VP is different from choosing VP then President. If order matters, use permutations.
- **Computing large factorials by hand.** Use a calculator! Factorials get huge very quickly.
- **Confusing  $P(n, k)$  with  $n^k$ .** Remember:  $P(n, k)$  is for no repetition,  $n^k$  is for repetition allowed.

## Classwork

- Calculate:
  - $5!$
  - $8!$
  - $\frac{10!}{8!}$
  - $P(7, 3)$
- A password must be 4 characters long, using only lowercase letters (a–z).
  - How many passwords are possible if letters can repeat?
  - How many passwords are possible if no letter can repeat?
- In how many ways can 6 books be arranged on a shelf?
- A class of 20 students needs to choose a President, Secretary, and Treasurer (all different). How many ways can this be done?
- A phone number has 7 digits.
  - How many phone numbers are possible?
  - How many have no repeated digits?

## Classwork Solutions

- $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$
  - $8! = 40,320$
  - $\frac{10!}{8!} = \frac{10 \times 9 \times 8!}{8!} = 10 \times 9 = 90$
  - $P(7, 3) = 7 \times 6 \times 5 = 210$
- With repetition:  $26^4 = 456,976$
  - Without repetition:  $26 \times 25 \times 24 \times 23 = 358,800$
- $6! = 720$  ways
- $P(20, 3) = 20 \times 19 \times 18 = 6,840$  ways
- With repetition:  $10^7 = 10,000,000$
  - Without repetition:  $10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 = 604,800$

## Homework

- Calculate:
  - $6!$
  - $\frac{9!}{7!}$
  - $P(8, 4)$
  - $P(10, 2)$
- In a club of 30 members, they are selecting a President, Vice-President, and Treasurer (all must be different people). How many ways are there to do this?
- 10 people must form a circle for a dance. In how many ways can they arrange themselves?
- A group of 6 friends always dine at the same table with exactly 6 chairs. They decide to sit in a different arrangement each day.
  - How many different arrangements are possible?
  - Can they keep this up for a whole year without repeating?
  - M** Does your answer depend on the shape of the table?
- In how many ways can 6 people stand in a row if Alice and Bob must stand next to each other?
- How many ways are there to seat 15 students in a classroom with:
  - Exactly 15 chairs?
  - 25 chairs?
- How many distinct arrangements are there of the letters in LEVEL?
- How many 5-digit numbers have no repeated digits? (Remember: a number cannot start with 0.)
- In how many ways can 7 people sit in a row if three specific people must NOT all sit together? (They can be in pairs, just not all three adjacent.)  
*Hint: Use complement—total arrangements minus arrangements where all three ARE together.*
- M** How many factors of 5 are in the prime factorization of  $100!$ ? How many trailing zeros does  $100!$  have?  
*Hint: Count multiples of 5, 25, 125, ...*

## Quick Check Answers

1.  $26^2 = 676$
2.  $26^3 \times 10^3 = 17,576,000$
3.  $P(10, 3) = 10 \times 9 \times 8 = 720$
4.  $10 \times 9 \times 8 \times 7 = 5,040$
5.  $(6 - 1)! = 5! = 120$  ways
6.  $(5 - 1)! = 4! = 24$  ways (but see note: necklaces can flip, so it's actually  $24/2 = 12$  if flipping counts as the same)
7.  $\frac{4!}{2!} = \frac{24}{2} = 12$  arrangements
8.  $\frac{4!}{2! \cdot 2!} = \frac{24}{4} = 6$  arrangements
9.  $3! \times 2 = 12$  ways (treat the pair as one unit, then multiply by 2 for internal order)
10.  $4! - 12 = 24 - 12 = 12$  ways