MATH 5: HANDOUT 23

PROBABILITY III: COUNTING AND PERMUTATIONS (SUMMARY)

Counting with Repetition Allowed

When making a sequence of choices, multiply the number of options at each step.

Theorem 1 (Multiplication Principle). If you choose k items from n options, with repetition allowed:

$$\text{Number of ways} = n^k$$

Example: 3-letter words from 26 letters?

$$26^3 = 17{,}576$$

Example: 4-digit PIN codes (0–9)?

$$10^4 = 10{,}000$$

Factorials

Definition 1 (Factorial). $n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$ (by convention, $0! = 1$)

Factorials grow fast: $5! = 120$, $10! \approx 3.6$ million, $20! \approx 2.4$ quintillion
To arrange all n items in order: $n!$ ways.

Counting Without Repetition (Permutations)

When each item can only be used once, the options decrease at each step.

Definition 2 (Permutations). Choosing k items from n options, **without repetition**, order matters:

$$P(n,k) = n \times (n-1) \times (n-2) \times \cdots \times (n-k+1) = \frac{n!}{(n-k)!}$$

Example: Gold, silver, bronze medals for 10 runners? $P(10,3) = 10 \times 9 \times 8 = 720$

Summary: With vs. Without Repetition

	With Repetition	Without Repetition
Rule	Can reuse items	Each item used at most once
Formula	n^k	$P(n,k) = n(n-1)(n-2)\cdots$
Example	PIN codes, passwords	Medals, officer positions

Circular Arrangements

At a round table, rotations of the same seating are identical. Fix one person's position and arrange the rest.

Theorem 2 (Circular Arrangements). The number of ways to arrange n distinct objects in a circle is $(n-1)!$

Example: 4 people around a round table? Fix one person, arrange the other 3: $(4-1)! = 3! = 6$ ways.

Arrangements with Identical Items

If some items are identical, we overcount by treating them as different. Divide out the repetition.

Theorem 3 (Permutations with Identical Items)**.** If you have n items where n_1 are identical of one type, n_2 of another, etc.:

$$\frac{n!}{n_1! \cdot n_2! \cdot n_3! \cdots}$$

Example: Arrangements of BANANA? 6 letters (A×3, N×2, B×1): $\frac{6!}{3! \cdot 2! \cdot 1!} = \frac{720}{12} = 60$

Counting with Restrictions

Items that must be adjacent: Treat them as a single "super-item," then multiply by the number of internal arrangements.

Example: 5 people in a row, A and B must be adjacent? Treat [AB] as one unit, arrange 4 objects: $4! \times 2 = 48$.

Items that must NOT be adjacent: Use the complement (total − adjacent).

Example: 5 people in a row, A and B *not* adjacent? $5! - 48 = 72$.

Common Mistakes

- **Using n^k when repetition is not allowed.** If items can't repeat, use permutations.
- **Confusing $P(n, k)$ with n^k.** $P(n, k)$ = no repetition; n^k = repetition allowed.
- **Forgetting whether order matters.** President then VP ≠ VP then President.
- **Row vs. circle.** A row of n has $n!$ arrangements; a circle has $(n-1)!$

Homework

1. Calculate:
   (a) $6!$
   (b) $\frac{9!}{7!}$
   (c) $P(8, 4)$
   (d) $P(10, 2)$
2. In a club of 30 members, they are selecting a President, Vice-President, and Treasurer (all must be different people). How many ways are there to do this?
3. 10 people must form a circle for a dance. In how many ways can they arrange themselves?
4. A group of 6 friends always dine at the same table with exactly 6 chairs. They decide to sit in a different arrangement each day.
   (a) How many different arrangements are possible?
   (b) Can they keep this up for a whole year without repeating?
   (c) M Does your answer depend on the shape of the table?
5. In how many ways can 6 people stand in a row if Alice and Bob must stand next to each other?
6. How many ways are there to seat 15 students in a classroom with:
   (a) Exactly 15 chairs?
   (b) 25 chairs?
7. How many distinct arrangements are there of the letters in LEVEL?
8. How many 5-digit numbers have no repeated digits? (Remember: a number cannot start with 0.)
9. In how many ways can 7 people sit in a row if three specific people must NOT all sit together? (They can be in pairs, just not all three adjacent.)
   Hint: Use complement—total arrangements minus arrangements where all three ARE together.
10. M How many factors of 5 are in the prime factorization of $100!$? How many trailing zeros does $100!$ have?
    Hint: Count multiples of 5, 25, 125, ...