MATH 5: HANDOUT 22

PROBABILITY II: THE PRODUCT RULE

The Product Rule

When we perform two experiments *independently* (the outcome of one doesn't affect the other), we multiply probabilities to find the probability that both occur.

Theorem

Product Rule (Multiplication Rule).
If two events A and B are *independent*, then:

$$P(A \text{ and } B) = P(A) \times P(B)$$

Example 1. You roll two dice. What is the probability of rolling a 5 on the first die AND a 6 on the second die?

The dice don't affect each other, so:

$$P(5 \text{ and } 6) = P(5) \times P(6) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$$

You toss a coin 3 times. What is the probability of getting heads all 3 times?

$$P(\text{HHH}) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

What Does "Independent" Mean?

Two events are **independent** if knowing the outcome of one event gives you *no information* about the other. Here's how to recognize independence:

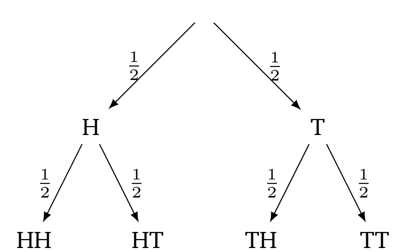
- **Independent:** Rolling two separate dice, flipping a coin multiple times, spinning a spinner twice. Each trial "starts fresh."
- **Not independent:** Drawing two cards *without* replacing the first one. After drawing an Ace, there are fewer Aces left, so the second draw is affected.

Warning!

The Product Rule only works for independent events! If events affect each other, you cannot simply multiply their probabilities. We'll learn how to handle dependent events in a later lesson.

Visualizing with Tree Diagrams

A **tree diagram** shows all possible outcomes and helps visualize the product rule. Each path through the tree represents one outcome, and we multiply probabilities along the branches.



To find $P(\text{HH})$, multiply along the path: $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.
Each of the four outcomes (HH, HT, TH, TT) has probability $\frac{1}{4}$, and they sum to 1.

Quick Check

1. You flip a coin twice. What is the probability of getting heads both times?
2. You roll a die and flip a coin. What is the probability of rolling a 6 and getting heads?

Multiple Independent Events

The product rule extends to any number of independent events:

$$P(A_1 \text{ and } A_2 \text{ and } \dots \text{ and } A_n) = P(A_1) \times P(A_2) \times \cdots \times P(A_n)$$

Example 2. If you toss a coin 10 times, what is the probability that all 10 are heads?

$$P(10 \text{ heads}) = \left(\frac{1}{2}\right)^{10} = \frac{1}{1024} \approx 0.001$$

This is about one-tenth of one percent!

Notice how quickly probabilities shrink when we require *all* events to succeed:

Coin flips	P(all heads)	Approximately
1	$\frac{1}{2}$	1 in 2
2	$\frac{1}{4}$	1 in 4
5	$\frac{1}{32}$	1 in 32
10	$\frac{1}{1024}$	1 in 1,000
20	$\frac{1}{1048576}$	1 in a million

This **exponential decay** is why long streaks are so rare—each additional requirement cuts the probability in half!

Quick Check

3. A die is rolled 3 times. What is the probability of rolling a 6 all three times?
4. A coin is tossed 4 times. What is the probability of getting tails every time?

"At Least One" Problems

Finding the probability of "at least one success" directly can be difficult because there are many ways it can happen. Instead, use the complement:

Theorem

At Least One Strategy.

$$P(\text{at least one success}) = 1 - P(\text{no successes})$$

Example 3. If you toss a coin 10 times, what is the probability of getting *at least one* heads?

Direct approach: We'd need to count 1 head, 2 heads, 3 heads, ..., 10 heads. Too complicated!

Complement approach:

$$\begin{aligned} P(\text{at least one heads}) &= 1 - P(\text{no heads}) \\ &= 1 - P(\text{all tails}) \\ &= 1 - \left(\frac{1}{2}\right)^{10} \\ &= 1 - \frac{1}{1024} = \frac{1023}{1024} \approx 0.999 \end{aligned}$$

Example 4. You roll a die 3 times. What is the probability of rolling *at least one* 6?

Method 1: Complement approach.

$$P(\text{at least one 6}) = 1 - P(\text{no 6s}) = 1 - \left(\frac{5}{6}\right)^3 = 1 - \frac{125}{216} = \frac{91}{216}$$

Method 2: Direct enumeration.
We count all outcomes with at least one 6 by considering separate cases:

- **Exactly three 6s:** Just $(6,6,6)$. Total: 1 outcome.
- **Exactly two 6s:** The non-6 can be 1, 2, 3, 4, or 5, and can appear in any of 3 positions:

$$\begin{array}{ll} (6,6,1),\ (6,6,2),\ (6,6,3),\ (6,6,4),\ (6,6,5) & \text{[6s in positions 1 \& 2]} \\ (6,1,6),\ (6,2,6),\ (6,3,6),\ (6,4,6),\ (6,5,6) & \text{[6s in positions 1 \& 3]} \\ (1,6,6),\ (2,6,6),\ (3,6,6),\ (4,6,6),\ (5,6,6) & \text{[6s in positions 2 \& 3]} \end{array}$$

  Total: $3 \times 5 = 15$ outcomes.

- **Exactly one 6:** The 6 can be in position 1, 2, or 3, and the other two rolls can each be 1, 2, 3, 4, or 5:

$$\begin{array}{ll} (6,1,1),\ (6,1,2),\ \ldots,\ (6,5,5) & \text{[25 outcomes with 6 in position 1]} \\ (1,6,1),\ (1,6,2),\ \ldots,\ (5,6,5) & \text{[25 outcomes with 6 in position 2]} \\ (1,1,6),\ (1,2,6),\ \ldots,\ (5,5,6) & \text{[25 outcomes with 6 in position 3]} \end{array}$$

  Total: $3 \times 25 = 75$ outcomes.

Adding up: $1 + 15 + 75 = 91$ favorable outcomes out of $6^3 = 216$ total.

$$P(\text{at least one 6}) = \frac{91}{216}$$

Both methods give the same answer! But the complement approach required just *one* simple calculation, while direct enumeration required counting three separate cases. The complement method is almost always easier for "at least one" problems.

Quick Check

5. You roll a die twice. What is the probability of getting at least one 6?
6. A coin is tossed 5 times. What is the probability of getting at least one tails?

Historical Note: The Problem That Started It All

Remember the Chevalier de Méré from Handout 21? He had another puzzle that led to the development of probability theory!
The Chevalier enjoyed two dice games:

- **Game A:** Roll one die 4 times. Win if you get *at least one 6*.
- **Game B:** Roll two dice 24 times. Win if you get *at least one double-6*.

He reasoned: "The chance of rolling a 6 is $\frac{1}{6}$, so in 4 rolls I should expect $\frac{4}{6}$ successes. The chance of double-6 is $\frac{1}{36}$, so in 24 rolls I should expect $\frac{24}{36} = \frac{4}{6}$ successes. The games should be equally good!"
But from experience, Game A seemed to win more often. Why?
The answer uses our "at least one" strategy:

$$P(\text{win Game A}) = 1 - P(\text{no 6 in 4 rolls}) = 1 - \left(\frac{5}{6}\right)^4 \approx 0.518$$

$$P(\text{win Game B}) = 1 - P(\text{no double-6 in 24 rolls}) = 1 - \left(\frac{35}{36}\right)^{24} \approx 0.491$$

Game A wins about 52% of the time, while Game B wins only about 49%! The Chevalier's intuition was wrong because you can't simply multiply probability by number of trials. Pascal and Fermat's mathematical analysis proved what the Chevalier had suspected from gambling experience.
Why this small difference matters. Both probabilities are close to 50%, so in a single evening of gambling, it would be hard to tell the games apart—sometimes you win, sometimes you lose. But there's a crucial difference:

- Game A has $P(\text{win}) > 50\%$. This is a **winning game**—if you play many times, you'll win more often than you lose.
- Game B has $P(\text{win}) < 50\%$. This is a **losing game**—if you play many times, you'll lose more often than you win.

Over hundreds of games, these small edges add up dramatically. If you bet $1 on each game:

- Playing Game A 1000 times, you'd expect to win about 518 and lose about 482, for a **profit** of roughly $36.
- Playing Game B 1000 times, you'd expect to win about 491 and lose about 509, for a **loss** of roughly $18.

This is exactly how casinos make money! Every casino game is designed so that $P(\text{casino wins}) > 50\%$—often by just a few percent. Players might win on any given night, but over millions of bets, the casino is guaranteed to profit. The lesson: in games of chance, even a tiny edge makes all the difference in the long run.

Warning!

The Gambler's Fallacy.
A common mistake is thinking that past outcomes affect future independent events. For example:
"I've flipped 5 heads in a row, so tails is due!"
This is **wrong**. Each coin flip is independent—the coin has no memory! The probability of heads on the next flip is still $\frac{1}{2}$, no matter what happened before.
Similarly, if a roulette wheel lands on red 10 times in a row, the next spin still has $P(\text{red}) = \frac{18}{37}$. The wheel doesn't "owe" you a black!

Probability as Percentages and Decimals

Probabilities can be expressed in three equivalent ways:

- As a **fraction**: $\frac{1}{4}$
- As a **decimal**: 0.25
- As a **percentage**: 25%

Conversions:

- Fraction to decimal: divide. Example: $\frac{3}{8} = 3 \div 8 = 0.375$
- Decimal to percentage: multiply by 100. Example: $0.375 = 37.5\%$
- Percentage to decimal: divide by 100. Example: $45\% = 0.45$

Example 5. Warning: Multiplying percentages.
The probability of winning a game is 5%. What is the probability of winning twice in a row?
Wrong: $5\% \times 5\% = 25\%$ (This treats $\%$ like a number!)
Correct: Convert to decimals first:

$$0.05 \times 0.05 = 0.0025 = 0.25\%$$

Quick Check

7. Convert $\frac{3}{5}$ to a percentage.
8. The probability of success is 20%. What is the probability of two successes in a row?

Key Takeaways

- For independent events: $P(A \text{ and } B) = P(A) \times P(B)$.
- For n independent events with the same probability p: $P(\text{all succeed}) = p^n$.
- For "at least one" problems: $P(\text{at least one}) = 1 - P(\text{none})$.
- When multiplying probabilities in percentages, convert to decimals first!

Common Mistakes

- **Using the product rule when events are not independent.** If drawing cards without replacement, the second draw depends on the first!
- **Multiplying percentages directly.** $5\% \times 5\% \neq 25\%$. Convert to decimals: $0.05 \times 0.05 = 0.0025 = 0.25\%$.
- **Forgetting about order.** Rolling a 5 then a 6 is different from rolling a 6 then a 5. If you want either order, you need to account for both.
- **Confusing "and" with "or."** Use multiplication for "and" (both happen), addition for "or" (at least one happens).

Classwork

1. You roll two dice. Find the probability of:
   (a) Rolling a 3 on the first die and a 5 on the second die
   (b) Rolling doubles (both dice show the same number)
   (c) Rolling a sum of 7
2. A coin is tossed 4 times. Find the probability of:
   (a) Getting HHHH
   (b) Getting HTHT (in that exact order)
   (c) Getting at least one heads
3. A basketball player makes 70% of her free throws. If she takes 2 free throws:
   (a) What is the probability she makes both?
   (b) What is the probability she misses both?
   (c) What is the probability she makes at least one?
4. A spinner has 4 equal sections: Red, Blue, Green, Yellow. If you spin twice:
   (a) What is the probability of getting Red both times?
   (b) What is the probability of getting the same color both times?
5. Convert each probability to the other two forms:
   (a) $\frac{1}{8}$
   (b) 0.6
   (c) 35%

Classwork Solutions

1. (a) $P(\text{3 and 5}) = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$
   (b) Doubles: (1,1), (2,2), (3,3), (4,4), (5,5), (6,6) = 6 outcomes. $P = \frac{6}{36} = \frac{1}{6}$
   (c) Sum of 7: (1,6), (2,5), (3,4), (4,3), (5,2), (6,1) = 6 outcomes. $P = \frac{6}{36} = \frac{1}{6}$
2. (a) $P(\text{HHHH}) = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$
   (b) $P(\text{HTHT}) = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$ (same as any specific sequence)
   (c) $P(\text{at least one H}) = 1 - P(\text{TTTT}) = 1 - \frac{1}{16} = \frac{15}{16}$
3. (a) $P(\text{both make}) = 0.70 \times 0.70 = 0.49 = 49\%$
   (b) $P(\text{both miss}) = 0.30 \times 0.30 = 0.09 = 9\%$
   (c) $P(\text{at least one}) = 1 - 0.09 = 0.91 = 91\%$
4. (a) $P(\text{Red twice}) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$
   (b) $P(\text{same color}) = \frac{4}{16} = \frac{1}{4}$ (can be RR, BB, GG, or YY)
5. (a) $\frac{1}{8} = 0.125 = 12.5\%$
   (b) $0.6 = \frac{3}{5} = 60\%$
   (c) $35\% = 0.35 = \frac{35}{100} = \frac{7}{20}$

Homework

Problems marked with M *or unmarked are expected from every student. Problems marked with* H *are optional challenge problems.*

1. A password consists of 2 letters followed by 3 digits. If each letter and digit is chosen randomly (26 letters, 10 digits), what is the probability that:

   (a) The password starts with "A"?

   (b) The password is exactly "AB123"?

   (c) All three digits are the same?

2. In roulette, there are 37 slots (0 through 36). Among 1–36, half are red and half are black (zero has no color). Find the probability of:

   (a) Getting red on a single spin

   (b) Getting red, then black, then 0 on three consecutive spins

   (c) Getting red 5 times in a row

3. A hunter shoots at a duck. The probability of hitting the duck with one shot is $\frac{1}{3}$.

   (a) What is the probability of missing the duck with one shot?

   (b) He makes 5 shots. What is the probability that he misses all five?

   (c) What is the probability that out of 5 shots, he hits at least once?

4. Supposing that boys and girls are equally likely to be born, what is the probability that:

   (a) The first 5 babies born at a hospital are all girls?

   (b) At least one of the first 5 babies is a girl?

5. M At a fair, you toss small balls into a crate full of bottles. Each ball has a 20% probability of landing inside a bottle. You win if at least one ball lands inside.

   (a) If you get 3 balls, what is the probability of winning?

   (b) If you get 5 balls, what is the probability of winning?

   (c) They charge $2 for 3 balls or $3 for 5 balls. Which is the better deal (considering only the probability of winning)?

6. M You draw two cards from a standard deck.

   (a) If you replace the first card before drawing the second (drawing **with replacement**), what is the probability that both cards are Aces?

   (b) If you do *not* replace the first card (drawing **without replacement**), what is the probability that both cards are Aces?

   (c) Which probability is higher? Explain why this makes sense.

   Compare this to Handout 21, Problem 5!

7. M Is the sequence RRRBR (red, red, red, black, red) in roulette more likely, less likely, or equally likely compared to the sequence BRRRR (black, red, red, red, red)? Explain your reasoning.

8. H You roll a die until you get a 6. What is the probability that:

   (a) You get a 6 on the first roll?

   (b) You get your first 6 on the second roll?

   (c) You get your first 6 on the third roll?

Quick Check Answers

1. $P(\text{HH}) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$
2. $P(\text{6 and H}) = \frac{1}{6} \times \frac{1}{2} = \frac{1}{12}$
3. $P(\text{three 6s}) = \left(\frac{1}{6}\right)^3 = \frac{1}{216}$
4. $P(\text{TTTT}) = \left(\frac{1}{2}\right)^4 = \frac{1}{16}$
5. $P(\text{at least one 6}) = 1 - P(\text{no 6s}) = 1 - \frac{5}{6} \times \frac{5}{6} = 1 - \frac{25}{36} = \frac{11}{36}$
6. $P(\text{at least one T}) = 1 - P(\text{all H}) = 1 - \frac{1}{32} = \frac{31}{32}$
7. $\frac{3}{5} = 0.6 = 60\%$
8. $0.20 \times 0.20 = 0.04 = 4\%$