MATH 5: HANDOUT 22

PROBABILITY II: THE PRODUCT RULE (SUMMARY)

The Product Rule

When two events are **independent** (one doesn't affect the other), multiply their probabilities:

Theorem 1 (Product Rule)**.** For independent events:

$$P(A \text{ and } B) = P(A) \times P(B)$$

For n independent events: $P(A_1 \text{ and } A_2 \text{ and } \cdots \text{ and } A_n) = P(A_1) \times P(A_2) \times \cdots \times P(A_n)$

Example: Roll two dice. $P(5 \text{ and } 6)$?

$$P = \frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$$

Example: Toss a coin 3 times. $P(\text{HHH})$?

$$P = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

What Does "Independent" Mean?

Events are **independent** if knowing one outcome gives no information about the other.

- **Independent:** Rolling dice, flipping coins, spinning spinners—each trial "starts fresh."
- **Not independent:** Drawing cards *without replacement*—fewer cards remain after each draw.

Probabilities Shrink Fast!

Coin flips	P(all heads)	Approx.
5	$\frac{1}{32}$	1 in 32
10	$\frac{1}{1024}$	1 in 1,000
20	$\frac{1}{1048576}$	1 in a million

"At Least One" Problems

Finding $P(\text{at least one success})$ directly is hard. Use the complement instead:

Theorem 2 (At Least One Strategy)**.** $P(\text{at least one success}) = 1 - P(\text{no successes})$

Example: Toss a coin 10 times. $P(\text{at least one heads})$?

$$P(\text{at least one H}) = 1 - P(\text{all tails}) = 1 - \left(\frac{1}{2}\right)^{10} = 1 - \frac{1}{1024} = \frac{1023}{1024} \approx 99.9\%$$

The Chevalier de Méré's Dice Problem

The Chevalier de Méré (from Handout 21) enjoyed two dice games:

- **Game A:** Roll one die 4 times. Win if you get at least one 6.
- **Game B:** Roll two dice 24 times. Win if you get at least one double-6.

He thought: "$\frac{1}{6} \times 4 = \frac{4}{6}$ and $\frac{1}{36} \times 24 = \frac{24}{36} = \frac{4}{6}$, so both games should be equally good!" But Game A seemed to win more often. Why?

Using our strategy:

$$P(\text{win Game A}) = 1 - \left(\frac{5}{6}\right)^4 \approx 0.518 = 51.8\%$$

$$P(\text{win Game B}) = 1 - \left(\frac{35}{36}\right)^{24} \approx 0.491 = 49.1\%$$

Game A is slightly better! You can't simply multiply probability by number of trials.

Why this matters: Both probabilities are close to 50%, so on any given night, it's hard to tell them apart. But Game A has $P > 50\%$ (a **winning game**—you profit over time), while Game B has $P < 50\%$ (a **losing game**—you lose over time). This is how casinos work: every game has $P(\text{casino wins}) > 50\%$ by just a few percent, guaranteeing profit over millions of bets.

The Gambler's Fallacy. Past outcomes don't affect future independent events. "I've flipped 5 heads in a row, so tails is due!" is **wrong**—each flip is still $\frac{1}{2}$. The coin has no memory!

Probability as Percentages

Probabilities can be written as fractions ($\frac{1}{4}$), decimals (0.25), or percentages (25%).

Conversions: Fraction $\to$ decimal: divide. Decimal $\to$ percent: $\times 100$. Percent $\to$ decimal: $\div 100$.

Warning: When multiplying probabilities given as percentages, convert to decimals first!
$P(\text{win}) = 5\%$. What is $P(\text{win twice in a row})$?
Wrong: $5\% \times 5\% = 25\%$ **Correct:** $0.05 \times 0.05 = 0.0025 = 0.25\%$

Common Mistakes to Avoid

- **Using product rule for dependent events:** Drawing cards without replacement is NOT independent!
- **Multiplying percentages directly:** $5\% \times 5\% \neq 25\%$. Convert to decimals first.
- **Confusing "and" with "or":** Use multiplication for "and," addition for "or."

Homework

Problems marked with M *or unmarked are expected from every student. Problems marked with* H *are optional challenge problems.*

1. A password consists of 2 letters followed by 3 digits. If each letter and digit is chosen randomly (26 letters, 10 digits), what is the probability that:
   (a) The password starts with "A"?
   (b) The password is exactly "AB123"?
   (c) All three digits are the same?
2. In roulette, there are 37 slots (0 through 36). Among 1–36, half are red and half are black (zero has no color). Find the probability of:
   (a) Getting red on a single spin
   (b) Getting red, then black, then 0 on three consecutive spins
   (c) Getting red 5 times in a row
3. A hunter shoots at a duck. The probability of hitting the duck with one shot is $\frac{1}{3}$.

(a) What is the probability of missing the duck with one shot?

(b) He makes 5 shots. What is the probability that he misses all five?

(c) What is the probability that out of 5 shots, he hits at least once?

4. Supposing that boys and girls are equally likely to be born, what is the probability that:

   (a) The first 5 babies born at a hospital are all girls?

   (b) At least one of the first 5 babies is a girl?

5. M At a fair, you toss small balls into a crate full of bottles. Each ball has a 20% probability of landing inside a bottle. You win if at least one ball lands inside.

   (a) If you get 3 balls, what is the probability of winning?

   (b) If you get 5 balls, what is the probability of winning?

   (c) They charge $2 for 3 balls or $3 for 5 balls. Which is the better deal (considering only the probability of winning)?

6. M You draw two cards from a standard deck.

   (a) If you replace the first card before drawing the second (drawing **with replacement**), what is the probability that both cards are Aces?

   (b) If you do *not* replace the first card (drawing **without replacement**), what is the probability that both cards are Aces?

   (c) Which probability is higher? Explain why this makes sense.

   Compare this to Handout 21, Problem 5!

7. M Is the sequence RRRBR (red, red, red, black, red) in roulette more likely, less likely, or equally likely compared to the sequence BRRRR (black, red, red, red, red)? Explain your reasoning.

8. H You roll a die until you get a 6. What is the probability that:

   (a) You get a 6 on the first roll?

   (b) You get your first 6 on the second roll?

   (c) You get your first 6 on the third roll?