MATH 5: HANDOUT 21

Probability I: Introduction

What is Probability?

Probability is the mathematics of chance. We use it to answer questions like: “What are the chances of rolling a 6?” or “How likely is it to rain tomorrow?”

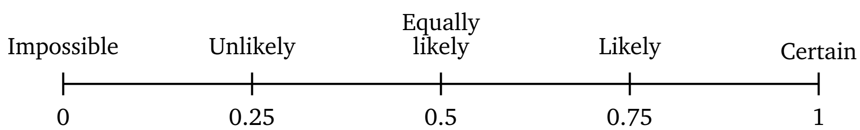
We study **experiments** (also called “tests” or “trials”) that have several possible **outcomes**. For example:

- Tossing a coin: outcomes are Heads (H) or Tails (T)
- Rolling a die: outcomes are 1, 2, 3, 4, 5, or 6
- Drawing a card: outcomes are any of the 52 cards

Definition

The **probability** of an outcome is a number between 0 and 1 that measures how likely it is to occur.

- Probability 0 means the outcome is *impossible*.
- Probability 1 means the outcome is *certain*.



When all outcomes are **equally likely**, we can calculate probability using the formula:

Theorem

Basic Probability Formula.

$$P(A) = \frac{\text{number of outcomes giving } A}{\text{total number of possible outcomes}}$$

Example 1. What is the probability of rolling a 3 on a fair die?

There are 6 equally likely outcomes (1, 2, 3, 4, 5, 6), and exactly 1 gives us a 3.

$$P(\text{rolling a } 3) = \frac{1}{6}$$

What is the probability of rolling an even number?

There are 3 even numbers (2, 4, 6) out of 6 possible outcomes.

$$P(\text{even}) = \frac{3}{6} = \frac{1}{2}$$

Quick Check

1. A bag contains 5 red marbles and 3 blue marbles. If you pick one marble at random, what is the probability of getting a red marble?
2. What is the probability of rolling a number greater than 4 on a die?

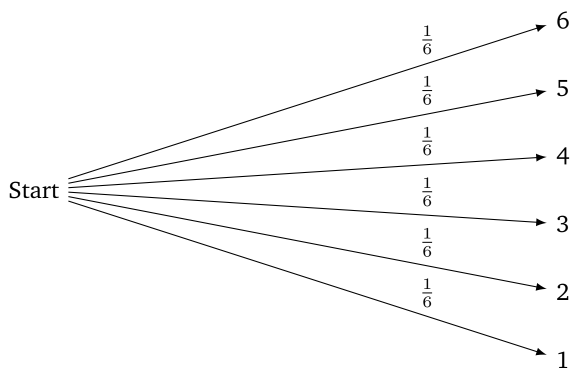
Historical Note: The Birth of Probability Theory

Probability theory was born in 1654 through letters between two French mathematicians: **Blaise Pascal** and **Pierre de Fermat**.
It started with a gambling question from a nobleman called the Chevalier de Méré: *"If two players must end a game early, how should they fairly divide the stakes based on their current scores?"* This was called the **Problem of Points**.
Pascal wrote to Fermat, and through their correspondence, they developed methods for calculating probabilities systematically. Their work showed that *chance* could be studied with the same mathematical rigor as geometry — a revolutionary idea at the time!

Blaise Pascal (left) and Pierre de Fermat (right)

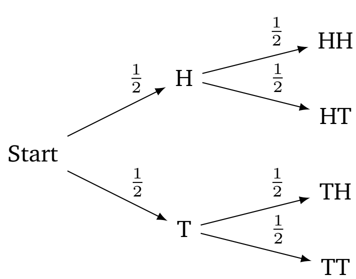
Tree Diagrams

A **tree diagram** is a visual way to show all possible outcomes of an experiment. Each "branch" represents one possible outcome.



Tree diagram for rolling a die

Tree diagrams are especially useful when an experiment has multiple stages (like flipping a coin twice):



Flipping a coin twice: 4 outcomes

Theoretical vs. Experimental Probability

There are two ways to find probability:

- **Theoretical probability**: Calculate using the formula $P = \frac{\text{favorable}}{\text{total}}$. This is what we expect *in theory*.

- **Experimental probability**: Perform the experiment many times and observe the results:

$$P_{\text{experimental}} = \frac{\text{number of times event occurred}}{\text{total number of trials}}$$

Example 2. The theoretical probability of heads when flipping a fair coin is $\frac{1}{2} = 50\%$.

If you flip a coin 100 times and get 47 heads, the experimental probability is $\frac{47}{100} = 47\%$.

As you do more trials, experimental probability gets closer to theoretical probability. This is called the **Law of Large Numbers**.

Quick Check

3. You flip a coin 80 times and get 52 heads. What is the experimental probability of heads? How does this compare to the theoretical probability?

Real-World Connection: Weather Forecasts and Probability

You've probably heard a weather forecast say something like "There's a 30% chance of rain tomorrow." But what does this actually mean? This is a perfect example of **experimental probability** in action!

What it means. When meteorologists say there's a 30% chance of rain, they mean: *"Out of 100 days with weather conditions just like tomorrow's, about 30 of them had rain."*

This probability comes from two sources:

- **Historical data**: Looking at past days with similar temperature, humidity, pressure, and wind patterns.
- **Computer models**: Running weather simulations many times with slightly different starting conditions. If 30 out of 100 runs predict rain, that's a 30% chance.

What it does NOT mean. Many people misunderstand rain probability! A 30% chance of rain does **not** mean:

- It will rain for 30% of the day ✗
- 30% of the city will get rain ✗
- It will "kind of" rain or drizzle ✗

It **does** mean: at any given location in the forecast area, there is a 30% chance of receiving measurable rainfall. Either it rains or it doesn't — the 30% tells you how likely rain is.

The Law of Large Numbers in action. How do we know weather forecasts are accurate? We check them over many days!

If a meteorologist predicts "30% chance of rain" on 100 different days, we'd expect it to actually rain on about 30 of those days. Weather services track this carefully. A *reliable* forecast means that when they say 30%, it really does rain about 30% of the time — not 10% or 50%.

This is exactly the Law of Large Numbers: over many trials, the experimental probability (how often it actually rained) gets closer to the predicted probability (30%).

Try this! Next time you see a weather forecast, think about what the probability really means. If it says 80% chance of rain, should you bring an umbrella? (Yes!) What about 20%? That's your call — but now you know exactly what you're risking!

The Addition Rule

Sometimes we want to find the probability that *one event or another* occurs.

Theorem

Addition Rule (for mutually exclusive events).
If events A and B cannot happen at the same time, then:

$$P(A \text{ or } B) = P(A) + P(B)$$

Example 3. What is the probability of drawing a King or a Queen from a standard deck of 52 cards?
Since a card cannot be both a King and a Queen at the same time, these events are *mutually exclusive*.

$$P(\text{King or Queen}) = P(\text{King}) + P(\text{Queen}) = \frac{4}{52} + \frac{4}{52} = \frac{8}{52} = \frac{2}{13}$$

Warning: Overlapping events. The addition rule only works when events cannot happen together. If they *can* overlap, we must be careful not to count outcomes twice.

Example 4. What is the probability of drawing a red card or a Queen?
Wrong approach: $P(\text{red}) + P(\text{Queen}) = \frac{26}{52} + \frac{4}{52} = \frac{30}{52}$
This is wrong because we counted the *red Queens* twice!
Correct approach: There are 26 red cards plus 2 black Queens = 28 favorable cards.

$$P(\text{red or Queen}) = \frac{28}{52} = \frac{7}{13}$$

For overlapping events, we can use the **General Addition Rule**:

Theorem

General Addition Rule.

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

We subtract $P(A \text{ and } B)$ to avoid counting the overlap twice.

Using this formula for the example above: $P(\text{red or Queen}) = \frac{26}{52} + \frac{4}{52} - \frac{2}{52} = \frac{28}{52} = \frac{7}{13}$.

Quick Check

4. What is the probability of rolling a 1 or a 6 on a die?
5. A bag has 4 red, 3 blue, and 5 green marbles. What is the probability of drawing a red or blue marble?

The Complement Rule

Sometimes it's easier to calculate the probability that something does *not* happen.

Theorem

Complement Rule.
The probability that event A does *not* happen is:

$$P(\text{not } A) = 1 - P(A)$$

Example 5. The probability of drawing a Queen is $\frac{4}{52} = \frac{1}{13}$.
The probability of drawing something *other than* a Queen is:

$$P(\text{not Queen}) = 1 - \frac{1}{13} = \frac{12}{13}$$

The probability of rolling a 6 on a die is $\frac{1}{6}$.
The probability of *not* rolling a 6 is:

$$P(\text{not } 6) = 1 - \frac{1}{6} = \frac{5}{6}$$

Finding the Complement of a Compound Event

For simple events, complements are easy: the complement of "Queen" is "not a Queen." But what about compound events like "red or Queen"? Let's think carefully about what the complement means.

Example 6. Find the complement of the event "red or Queen."
Remember: the complement contains everything that is **not** in the original event.
A card is "red or Queen" if it satisfies *at least one* of these conditions:

- It is red, OR
- It is a Queen (or both!)

So a card is **not** "red or Queen" if it fails *both* conditions:

- It is not red (so it must be black), AND
- It is not a Queen

Therefore: **the complement of "red or Queen" is "black and not a Queen."**
In other words, the complement consists of all black non-Queen cards.

This leads to an important principle:

*The complement of "A or B" is "not A **and** not B."*

Now let's use this to calculate a probability:

Example 7. What is $P(\text{black and not a Queen})$?
Method 1: Direct counting.
There are 26 black cards. Two of them are Queens (Q♠, Q♣).
So there are $26 - 2 = 24$ black non-Queens.

$$P(\text{black and not Queen}) = \frac{24}{52} = \frac{6}{13}$$

Method 2: Using the complement rule.
We already found that $P(\text{red or Queen}) = \frac{28}{52} = \frac{7}{13}$.
Since "black and not Queen" is the complement of "red or Queen":

$$P(\text{black and not Queen}) = 1 - P(\text{red or Queen}) = 1 - \frac{7}{13} = \frac{6}{13} \quad \checkmark$$

Quick Check

6. If the probability of rain tomorrow is $\frac{2}{5}$, what is the probability it will not rain?
7. The probability of drawing an Ace from a deck is $\frac{1}{13}$. What is the probability of drawing a card that is not an Ace?

Key Takeaways

- Probability measures how likely an event is to occur, on a scale from 0 (impossible) to 1 (certain).
- When outcomes are equally likely: $P(A) = \frac{\text{favorable outcomes}}{\text{total outcomes}}$.
- Addition Rule: $P(A \text{ or } B) = P(A) + P(B)$ when A and B cannot both happen.
- Complement Rule: $P(\text{not } A) = 1 - P(A)$.

Common Mistakes

- **Adding probabilities when events overlap.** If events can happen together, you'll count some outcomes twice. Only use the addition rule for mutually exclusive events.
- **Forgetting that probabilities must be between 0 and 1.** If your answer is negative or greater than 1, something went wrong.
- **Confusing "at least one" with "exactly one."** These are different! "At least one" includes one, two, three, etc.
- **Assuming outcomes are equally likely when they're not.** A bent coin might not have equal chances for heads and tails.

Classwork

1. A spinner has 8 equal sections numbered 1 through 8. Find the probability of:
   (a) Spinning a 5
   (b) Spinning an odd number
   (c) Spinning a number less than 3
2. A bag contains 6 red balls, 4 blue balls, and 2 green balls. If you pick one ball at random, find:
   (a) $P(\text{red})$
   (b) $P(\text{blue or green})$
   (c) $P(\text{not green})$
3. From a standard deck of 52 cards, find the probability of drawing:
   (a) A heart
   (b) A face card (Jack, Queen, or King)
   (c) A card that is not a spade
4. You roll a standard die. What is the probability of rolling:
   (a) A number divisible by 3
   (b) A number greater than 2
   (c) A 7
5. The probability that a student passes a test is 0.85. What is the probability that the student fails?

Classwork Solutions

1. (a) $P(5) = \frac{1}{8}$
   (b) Odd numbers: 1, 3, 5, 7 (4 numbers). $P(\text{odd}) = \frac{4}{8} = \frac{1}{2}$
   (c) Numbers less than 3: 1, 2. $P(\text{less than 3}) = \frac{2}{8} = \frac{1}{4}$
2. Total balls = $6 + 4 + 2 = 12$.
   (a) $P(\text{red}) = \frac{6}{12} = \frac{1}{2}$
   (b) $P(\text{blue or green}) = \frac{4+2}{12} = \frac{6}{12} = \frac{1}{2}$
   (c) $P(\text{not green}) = 1 - \frac{2}{12} = \frac{10}{12} = \frac{5}{6}$
3. (a) $P(\text{heart}) = \frac{13}{52} = \frac{1}{4}$
   (b) Face cards: 4 Jacks + 4 Queens + 4 Kings = 12. $P(\text{face card}) = \frac{12}{52} = \frac{3}{13}$
   (c) $P(\text{not spade}) = 1 - \frac{13}{52} = \frac{39}{52} = \frac{3}{4}$
4. (a) Divisible by 3: 3, 6. $P = \frac{2}{6} = \frac{1}{3}$
   (b) Greater than 2: 3, 4, 5, 6. $P = \frac{4}{6} = \frac{2}{3}$
   (c) A die has no 7. $P(7) = 0$
5. $P(\text{fail}) = 1 - 0.85 = 0.15$

Homework

Problems marked with M *or unmarked are expected from every student. Problems marked with* H *are optional challenge problems.*

1. A jar contains 10 red candies, 7 blue candies, and 3 yellow candies. If you pick one candy at random, find:
   (a) The probability of getting a blue candy
   (b) The probability of getting a red or yellow candy
   (c) The probability of not getting a red candy
2. In the game of roulette, there are 37 slots numbered 0 through 36. Of numbers 1–36, half are red and half are black (zero has no color). Find the probability of:
   (a) Hitting a number between 1 and 12 (inclusive)
   (b) Hitting an even number other than zero
   (c) Hitting a red number or zero
3. From a standard deck of 52 cards, find the probability of drawing:
   (a) The Queen of Spades
   (b) A black King
   (c) Anything except the Queen of Hearts
4. M You roll two dice (one red, one black).
   (a) How many total outcomes are possible?
   (b) What is the probability of rolling two 1s (snake eyes)?
   (c) What is the probability of rolling a 4 on the red die and a 6 on the black die?
5. M I draw a card from a deck and it turns out to be an Ace. Now I draw another card from the same deck (without replacing the first card). What is the probability that the second card is also an Ace?
6. What is the probability that a randomly chosen person was born:
   (a) In January? (Assume all months are equally likely)
   (b) On February 5? (Assume all 365 days are equally likely)
   (c) On a Sunday? (Assume all days of the week are equally likely)
7. M A box contains 500 candies of different colors and sizes. There are 100 large candies and 400 small ones. There are 70 red candies, of which 11 are large. What is the probability that a randomly chosen candy is either red or large (or both)?
8. M Using the same box of candies from Problem 7:
   (a) What is the complement of the event "red or large"? Describe it in words.
   (b) Find the probability of this complement by counting directly.
   (c) Find the probability of this complement using the complement rule and your answer to Problem 7. Verify that you get the same answer.
9. M You roll two dice. What is the probability that the product of the two numbers is a multiple of 3?
10. M If you toss a coin 5 times:

(a) What is the probability that all 5 tosses are heads?
(b) What is the probability of getting the exact sequence HHTHT?
(c) Which is more likely: all heads, or the sequence HHTHT?

11. **H** You roll a die 3 times. What is the probability of rolling at least one 6?

12. **H** A family has two children. Given that at least one child is a boy, what is the probability that both children are boys? *(Hint: List all equally likely outcomes for two children, then exclude impossible cases.)*

Quick Check Answers

1. $P(\text{red}) = \frac{5}{8}$
2. Numbers greater than 4: 5, 6. $P = \frac{2}{6} = \frac{1}{3}$
3. $P_{\text{exp}}(\text{heads}) = \frac{52}{80} = \frac{13}{20} = 0.65 = 65\%$. This is higher than the theoretical probability of 50%.
4. $P(1 \text{ or } 6) = \frac{1}{6} + \frac{1}{6} = \frac{2}{6} = \frac{1}{3}$
5. $P(\text{red or blue}) = \frac{4+3}{12} = \frac{7}{12}$
6. $P(\text{no rain}) = 1 - \frac{2}{5} = \frac{3}{5}$
7. $P(\text{not Ace}) = 1 - \frac{1}{13} = \frac{12}{13}$