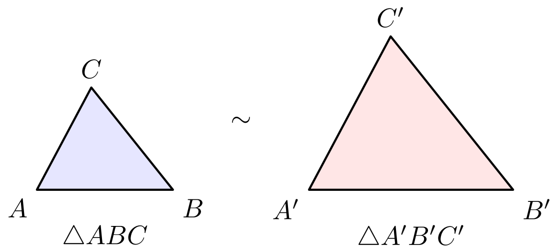
MATH 5: HANDOUT 20

GEOMETRY VII: SIMILAR TRIANGLES

Similar Figures

Two figures are called **similar** if one of them can be obtained from the other by *rescaling* (stretching or shrinking equally in all directions). We use the symbol $\sim$ to denote similarity.



In similar figures:

- All corresponding **angles** are equal.
- All corresponding **lengths** are in the same ratio.

For example, if $\triangle ABC \sim \triangle A'B'C'$, then:

$$\angle A = \angle A', \quad \angle B = \angle B', \quad \angle C = \angle C'$$

and there exists a number k (called the **similarity ratio** or **scale factor**) such that:

$$A'B' = k \cdot AB, \quad A'C' = k \cdot AC, \quad B'C' = k \cdot BC.$$

Example 1. If $\triangle ABC \sim \triangle DEF$ with scale factor $k = 2$, and $AB = 3$, $BC = 4$, $AC = 5$, then:

$$DE = 6, \quad EF = 8, \quad DF = 10.$$

The larger triangle has sides exactly twice as long as the smaller one.

Quick Check

1. Two similar triangles have a scale factor of 3. If a side of the smaller triangle is 5 cm, what is the corresponding side of the larger triangle?
2. In similar triangles, are the angles equal or proportional?

The AAA Similarity Test

How can we tell if two triangles are similar? Unlike congruence (where we need SSS, SAS, or ASA), similarity is much easier to check:

Theorem

AAA Similarity Rule. If three angles of one triangle are equal to the corresponding angles of another triangle, then the triangles are similar.

In fact, since the angles of a triangle sum to $180°$, we only need to check **two angles**:

Theorem

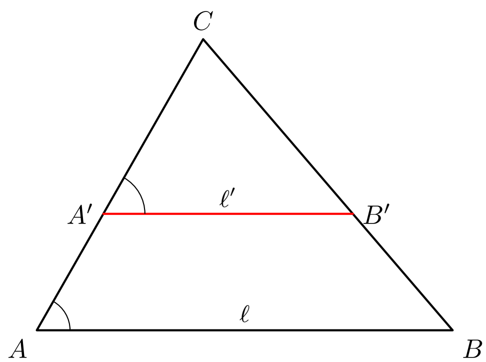
AA Similarity Rule. If two angles of one triangle are equal to two angles of another triangle, then the triangles are similar.

Example 2. If $\triangle ABC$ and $\triangle DEF$ are both right triangles with $\angle C = \angle F = 90°$, and $\angle A = \angle D = 30°$, then $\triangle ABC \sim \triangle DEF$.

Note: There is also an **SSS Similarity Rule**: if all three pairs of corresponding sides are proportional (i.e., $\frac{AB}{DE} = \frac{BC}{EF} = \frac{CA}{FD}$), then the triangles are similar. However, the AA rule is usually easier to apply.

Parallel Lines Create Similar Triangles

One of the most common ways similar triangles appear is when a line parallel to one side of a triangle cuts through the other two sides.



Theorem

Theorem. Let $\triangle ABC$ be a triangle, and let line $A'B'$ be parallel to side AB. Then $\triangle A'B'C \sim \triangle ABC$.

Proof. Since $A'B' \parallel AB$, by the theorem on alternate interior angles:

- $\angle CA'B' = \angle CAB$ (corresponding angles)
- $\angle CB'A' = \angle CBA$ (corresponding angles)

The angle at C is shared by both triangles. Thus all three angles are equal, so by AAA, the triangles are similar. □

Quick Check

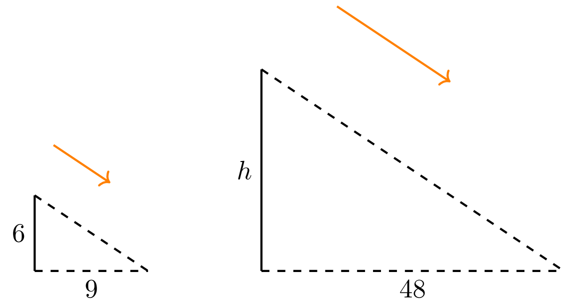
3. In the figure above, if $CA' = 2$, $A'A = 3$, and $AB = 10$, find $A'B'$.
4. Two right triangles both have a $45°$ angle. Are they similar? Why?

Using Similar Triangles

Similar triangles are incredibly useful for finding unknown lengths, especially when direct measurement is impossible.

Example 3. Finding a Height Using Shadows.
A 6-foot tall person casts a 9-foot shadow. At the same time, a tree casts a 48-foot shadow. How tall is the tree?

Solution. The sun's rays are parallel, so the triangles formed by the person and the tree (with their shadows) are similar.

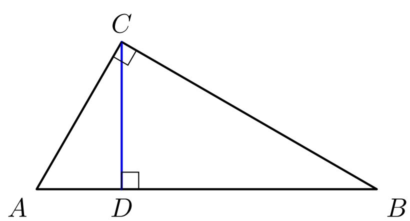


Let h be the height of the tree. Since the triangles are similar:

$$\frac{h}{6} = \frac{48}{9}$$

$$h = 6 \times \frac{48}{9} = \frac{288}{9} = 32 \text{ feet.}$$

Example 4. The Altitude to the Hypotenuse.
In a right triangle ABC with $\angle C = 90^\circ$, let CD be the altitude from C to the hypotenuse AB.



Then three similar triangles are formed:

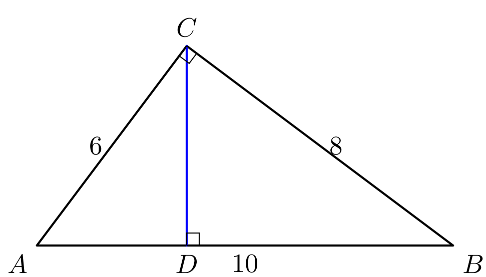
$$\triangle ABC \sim \triangle ACD \sim \triangle CBD.$$

Why? Each pair shares an angle (at A or B) and both have a right angle, so by AA they are similar.
This leads to useful relationships:

$$CD^2 = AD \cdot DB \quad \text{(altitude rule)}$$

$$AC^2 = AD \cdot AB, \quad BC^2 = BD \cdot AB \quad \text{(leg rules)}$$

Numeric Example. Let $AC = 6$, $BC = 8$. Then $AB = \sqrt{6^2 + 8^2} = 10$.



To find AD, use the similarity $\triangle ACD \sim \triangle ABC$:

$$\frac{AD}{AC} = \frac{AC}{AB} \implies AD = \frac{AC^2}{AB} = \frac{36}{10} = 3.6.$$

Similarly, from $\triangle BCD \sim \triangle ABC$:

$$\frac{BD}{BC} = \frac{BC}{AB} \implies BD = \frac{BC^2}{AB} = \frac{64}{10} = 6.4.$$

Check: $AD + BD = 3.6 + 6.4 = 10 = AB$. ✓
Finally, the altitude: $CD = \sqrt{AD \cdot BD} = \sqrt{3.6 \times 6.4} = \sqrt{23.04} = 4.8$.

Quick Check

5. A 2-meter stick casts a 3-meter shadow. A building casts a 45-meter shadow. How tall is the building?
6. In a right triangle, the altitude to the hypotenuse divides it into segments of length 4 and 9. Find the length of the altitude.

Scale Factor and Area

When figures are similar with scale factor k:

- Linear measurements (lengths, perimeters) are multiplied by k.
- Areas are multiplied by k^2.

Theorem

Theorem. If two similar figures have scale factor k, then the ratio of their areas is k^2.

Example 5. Two similar triangles have sides in the ratio $2 : 3$. What is the ratio of their areas?
Solution. The scale factor is $k = \frac{3}{2}$. The ratio of areas is:

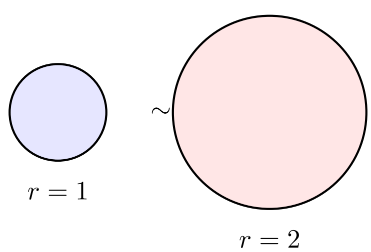
$$k^2 = \left(\frac{3}{2}\right)^2 = \frac{9}{4}.$$

So if the smaller triangle has area 8, the larger has area $8 \times \frac{9}{4} = 18$.

Other Similar Figures

While we have focused on triangles, similarity applies to all shapes. Two figures are similar if one can be obtained from the other by uniform scaling.

Circles are always similar. Any two circles are similar to each other—you can always scale one circle to match another. The scale factor is simply the ratio of their radii (or diameters).



Since circles are similar, the area relationship k^2 applies: if the radius doubles, the area quadruples!

Example 6. The Pizza Problem.
A pizzeria offers three sizes of pizza:

Size	Diameter	Price
Small	$8''$	$8
Medium	$12''$	$14
Large	$16''$	$18

Which pizza is the best deal (most pizza per dollar)?

Solution. Pizza is circular, so we need to compare areas, not diameters!
Using radius $= \frac{\text{diameter}}{2}$:

- Small: $r = 4$, area $= \pi(4)^2 = 16\pi$ square inches
- Medium: $r = 6$, area $= \pi(6)^2 = 36\pi$ square inches
- Large: $r = 8$, area $= \pi(8)^2 = 64\pi$ square inches

Now compute the price per square inch (we can ignore π since it appears in all):

- Small: $\frac{\$8}{16\pi} = \frac{\$0.50}{\pi}$ per sq. inch
- Medium: $\frac{\$14}{36\pi} \approx \frac{\$0.39}{\pi}$ per sq. inch
- Large: $\frac{\$18}{64\pi} \approx \frac{\$0.28}{\pi}$ per sq. inch

The **large pizza** is the best deal—you pay the least per square inch of pizza!

Another way to see this: The large pizza has diameter $\frac{16}{8} = 2$ times the small, so it has $2^2 = 4$ times the area. But it only costs $\frac{\$18}{\$8} = 2.25$ times as much. Four times the pizza for only 2.25 times the price!

Quick Check

7. A $10''$ pizza costs $12 and a $15''$ pizza costs $20. Which is the better deal?

Key Takeaways

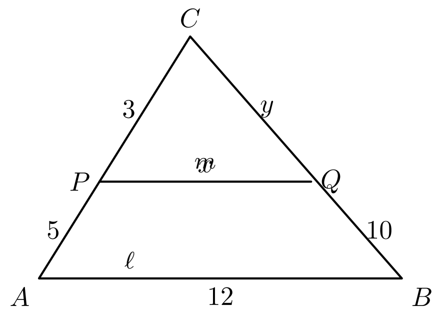
- Similar figures have equal angles and proportional sides.
- The AA (or AAA) rule: Two triangles are similar if two (or three) pairs of angles are equal.
- A line parallel to one side of a triangle creates a smaller similar triangle.
- Similar triangles can be used to find heights and distances indirectly.
- If the scale factor is k, then areas scale by k^2.
- All circles are similar to each other (scale factor = ratio of radii).

Common Mistakes

- **Confusing similar and congruent.** Similar means same shape (angles equal, sides proportional). Congruent means same shape AND same size.
- **Setting up proportions incorrectly.** Make sure corresponding sides are in the same position in each ratio. If $\triangle ABC \sim \triangle DEF$, then $\frac{AB}{DE} = \frac{BC}{EF}$, not $\frac{AB}{EF}$.
- **Using the scale factor for area.** If sides are in ratio k, areas are in ratio k^2, not k.
- **Forgetting AA is enough.** You don't need to check all three angles—two is sufficient (the third is determined by the $180°$ sum).

Classwork

1. Triangles ABC and DEF are similar with $\triangle ABC \sim \triangle DEF$. If $AB = 6$, $BC = 8$, $AC = 10$, and $DE = 9$, find EF and DF.
2. In the figure, lines ℓ and m are parallel. Find the lengths x and y.



3. A flagpole casts a shadow 24 feet long. At the same time, a 5-foot person standing nearby casts a shadow 8 feet long. How tall is the flagpole?
4. Two similar triangles have areas 16 cm² and 36 cm².
   (a) What is the ratio of their corresponding sides?
   (b) If the perimeter of the smaller triangle is 20 cm, what is the perimeter of the larger?
5. A small triangle has sides 3, 4, 5 and area 6 cm². A similar triangle has sides 6, 8, 10. What is the area of the larger triangle?
6. In right triangle ABC with $\angle C = 90^\circ$, $AC = 6$ and $BC = 8$. The altitude from C meets AB at D. Find CD, AD, and DB.

Classwork Solutions

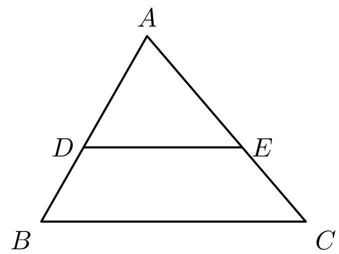
1. Scale factor $k = \frac{DE}{AB} = \frac{9}{6} = \frac{3}{2}$.
   $EF = BC \times \frac{3}{2} = 8 \times \frac{3}{2} = 12$.
   $DF = AC \times \frac{3}{2} = 10 \times \frac{3}{2} = 15$.
2. Since $PQ \parallel AB$, triangles CPQ and CAB are similar.
   Scale factor: $\frac{CP}{CA} = \frac{3}{3+5} = \frac{3}{8}$.
   $x = PQ = AB \times \frac{3}{8} = 12 \times \frac{3}{8} = 4.5$.
   For y: $\frac{CQ}{CB} = \frac{3}{8}$, so $\frac{y}{y+10} = \frac{3}{8}$.
   $8y = 3(y + 10) = 3y + 30$, so $5y = 30$, thus $y = 6$.
3. The triangles formed are similar (same sun angle).
   $\frac{\text{flagpole height}}{5} = \frac{24}{8} = 3$.
   Flagpole height $= 5 \times 3 = 15$ feet.
4. (a) Ratio of areas $= \frac{36}{16} = \frac{9}{4} = k^2$, so $k = \frac{3}{2}$.
   Ratio of sides $= \frac{3}{2}$ (or $3 : 2$).
   (b) Perimeter scales by k: larger perimeter $= 20 \times \frac{3}{2} = 30$ cm.
5. Scale factor $k = \frac{6}{3} = 2$. Area scales by $k^2 = 4$.
   Larger area $= 6 \times 4 = 24$ cm².

6. First find $AB = \sqrt{6^2 + 8^2} = \sqrt{100} = 10$.
   Using similar triangles: $\triangle ACD \sim \triangle ABC$, so $\frac{AD}{AC} = \frac{AC}{AB}$.
   $AD = \frac{AC^2}{AB} = \frac{36}{10} = 3.6$.
   $DB = AB - AD = 10 - 3.6 = 6.4$.
   $CD = \sqrt{AD \times DB} = \sqrt{3.6 \times 6.4} = \sqrt{23.04} = 4.8$.

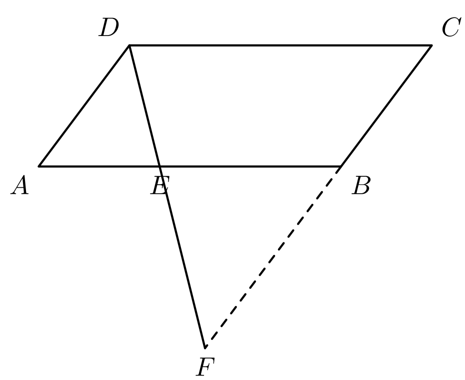
Homework

Problems marked with M *or unmarked are expected from every student. Problems marked with* H *are optional challenge problems.*

1. Triangles ABC and PQR are similar. If $AB = 4$, $BC = 5$, $CA = 6$, and $PQ = 10$, find QR and RP.
2. In the figure, $DE \parallel BC$. If $AD = 4$, $DB = 6$, and $BC = 15$, find DE.



3. M In the story *The Musgrave Ritual*, Sherlock Holmes needed to find the length of a shadow cast by a tree that no longer existed. He knew the tree had been 64 feet tall. To estimate the shadow, he set up a 6-foot rod and measured its shadow to be 9 feet at the same time of day. How long would the tree's shadow have been?
4. M In right triangle ABC with $\angle C = 90°$, $AC = 4$ cm and $AB = 5$ cm. The altitude CD is drawn to the hypotenuse.
   (a) Find BC.
   (b) Show that triangles ABC and ACD are similar.
   (c) Find AD and CD.
5. M Two similar triangles have corresponding sides in ratio $2 : 5$.
   (a) If a side of the smaller triangle is 7, what is the corresponding side of the larger?
   (b) If the area of the larger triangle is 100 cm^2, what is the area of the smaller?
6. M A 1.5-meter tall child stands 12 meters from a lamppost and casts a 3-meter shadow. How tall is the lamppost?
7. H In the figure, $ABCD$ is a parallelogram. E is on AB such that $AE : EB = 2 : 3$. Line DE extended meets CB extended at F. Find $BF : BC$.



8. H Prove that the line connecting the midpoints of two sides of a triangle is parallel to the third side and equal to half its length. (This is called the **Midsegment Theorem**.)

Quick Check Answers

1. $5 \times 3 = 15$ cm.
2. Equal (angles are preserved in similar figures).
3. Scale factor $= \frac{CA'}{CA} = \frac{2}{2+3} = \frac{2}{5}$. So $A'B' = 10 \times \frac{2}{5} = 4$.
4. Yes. They share a right angle and a $45°$ angle, so by AA they are similar.
5. $\frac{h}{2} = \frac{45}{3} = 15$, so $h = 30$ meters.
6. $CD = \sqrt{4 \times 9} = \sqrt{36} = 6$.
7. $10''$ pizza: area $= 25\pi$, cost per area $= \frac{12}{25\pi} = \frac{0.48}{\pi}$.
   $15''$ pizza: area $= 56.25\pi$, cost per area $= \frac{20}{56.25\pi} \approx \frac{0.36}{\pi}$. The $15''$ pizza is the better deal.