MATH 5: HANDOUT 19

GEOMETRY VI: STRAIGHTEDGE AND COMPASS CONSTRUCTIONS

Straightedge–and–Compass Constructions

In classical geometry we draw using only two tools: a **straightedge** (ruler without marks) to draw straight lines, and a **compass** to draw circles and copy lengths.

In this section we focus on the foundational constructions that form the building blocks for all others. Even complex constructions rely on the same underlying ideas: copying a length with a compass, intersecting circles and lines, forming isosceles triangles, and using triangle congruence to justify each step. With just these tools, an entire geometric universe becomes constructible.

Historical Background

For more than two thousand years, geometers have explored what can (and cannot) be constructed using only two simple tools: a *straightedge* for drawing lines and a *compass* for drawing circles and copying lengths. This tradition goes back to the ancient Greeks, especially Euclid's *Elements* (300 BC), which is built entirely on straightedge–and–compass constructions.

Despite the simplicity of these tools, they allow surprisingly powerful and beautiful constructions: one can erect perpendiculars, bisect any segment or angle, copy shapes, construct regular polygons like the equilateral triangle, square, pentagon, and even the stunning 17-gon discovered by Gauss when he was only 19 in 1796. On the other hand, several famous classical problems turned out to be *impossible* with these tools alone—such as trisecting an arbitrary angle, doubling the cube, or squaring the circle. The proofs of impossibility took over two millennia and required modern algebra to settle.

Carl Friedrich Gauss (1777-1855)

Constructing an Equilateral Triangle

Problem. Given a segment AB, construct an equilateral triangle on it (that is, a triangle $\triangle ABC$ with all three sides equal to AB).

Construction.

1. With center at A and radius AB, draw a circle.
2. With center at B and the same radius AB, draw a second circle.
3. Let the circles intersect at a point C. Then draw segments AC and BC. Triangle $\triangle ABC$ is equilateral.

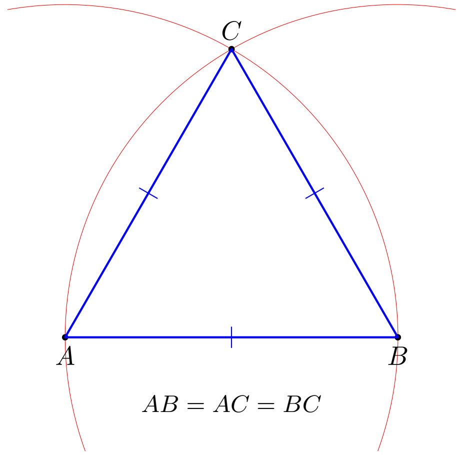
Why it works (reasoning). Point C lies on both circles, so

$$AC = AB \quad \text{and} \quad BC = AB.$$

Thus all three sides satisfy

$$AB = AC = BC,$$

which is exactly the definition of an equilateral triangle.



Constructing a Triangle from Three Given Sides (SSS)

Problem. Given segments of lengths a, b, c, construct a triangle with side lengths a, b, c (if possible).

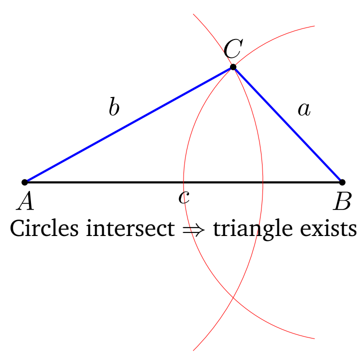
Construction.

1. Draw a segment AB with length c (choose a convenient scale; AB will be a base).
2. With the compass set to length b, draw a circle centered at A.
3. With the compass set to length a, draw a circle centered at B.
4. If the two circles intersect, choose one intersection and call it C. Then $\triangle ABC$ has sides $AB = c$, $AC = b$, $BC = a$.

Why it works & when it fails (Triangle Inequality). The two circles intersect *iff* the distance between centers $AB = a$ is *strictly less* than $b + c$ and also greater than $|b - c|$. In particular, if we let a be the longest given length, it is necessary and sufficient that

$$a < b + c.$$

If instead $a = b + c$, the circles are tangent and the "triangle" collapses into a straight line (degenerate). If $a > b + c$, the circles are disjoint and no triangle exists.



Example 1. **Try it.** Can you construct sides $2, 3, 6$? Here $6 \not< 2 + 3$; the circles do not meet ⇒ no triangle.

Quick Check

1. Can a triangle be constructed with sides 3, 4, 5? Why or why not?
2. Can a triangle be constructed with sides 1, 2, 5? Why or why not?
3. In an equilateral triangle, what are all three angles?

Copying (Replicating) an Angle

Problem. Given an angle $\angle AOB$, construct an angle at a different point O' that is congruent to it.

Construction. You are given $\angle AOB$, and you want to construct a congruent angle starting at O'.

1. Draw a ray $O'P$ starting at O'. This will correspond to the side OA.
2. With center at O, choose any convenient radius and draw an arc that intersects OA at C and OB at D.
3. Using the *same radius*, draw an arc centered at O', intersecting the ray $O'P$ at C'.
4. Measure the distance CD with the compass. With center at C' and radius CD, draw a circle; it intersects the previous arc at a point D'.
5. Draw the ray $O'D'$. Then $\angle C'O'D' \cong \angle AOB$.

Why it works (reasoning). From the first arc, points C and D satisfy

$$OC = OD,$$

and the length CD captures how "wide" the angle is along that arc.

We replicate the same structure at O':

$$O'C' = OC, \qquad C'D' = CD.$$

Thus triangles $\triangle OCD$ and $\triangle O'C'D'$ have:

$$OC = O'C', \qquad CD = C'D', \qquad OD = O'D'.$$

(The last equality holds because both OD and $O'D'$ equal the same chosen radius.)

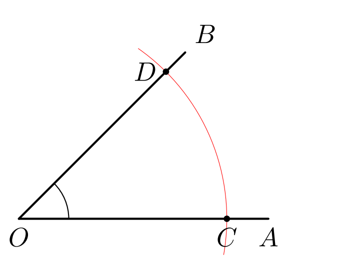
Therefore,

$$\triangle OCD \cong \triangle O'C'D' \quad (\text{by } \textbf{SSS}).$$

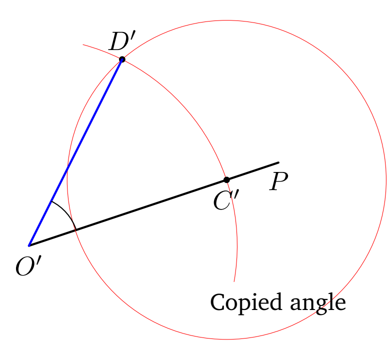
Corresponding angles are equal, in particular

$$\angle COD = \angle C'O'D',$$

so the constructed angle at O' is a perfect copy of the original.



Original angle



Copied angle

Bisecting a Segment (Perpendicular Bisector)

Problem. Given a segment AB, construct its midpoint M (so that $AM = MB$).

Construction.

1. With radius $r > \frac{1}{2}AB$, draw a circle centered at A.
2. With the same radius r, draw a circle centered at B.
3. Let the two circles intersect at P and Q. Draw line PQ.
4. The line PQ is the *perpendicular bisector* of AB; set $M = PQ \cap AB$.

Why it works (reasoning). Points P and Q lie on both circles, so

$$AP = BP = r \quad \text{and} \quad AQ = BQ = r.$$

Thus triangles APQ and BPQ have all three corresponding sides equal, so

$$\triangle APQ \cong \triangle BPQ \quad \text{(by SSS)}.$$

From the congruence, the angles at P in the two triangles are equal:

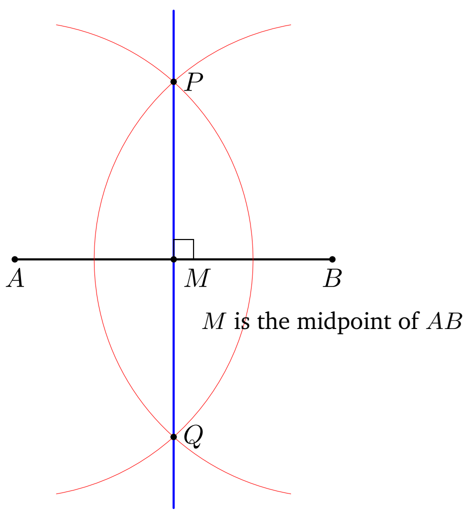
$$\angle APQ = \angle BPQ.$$

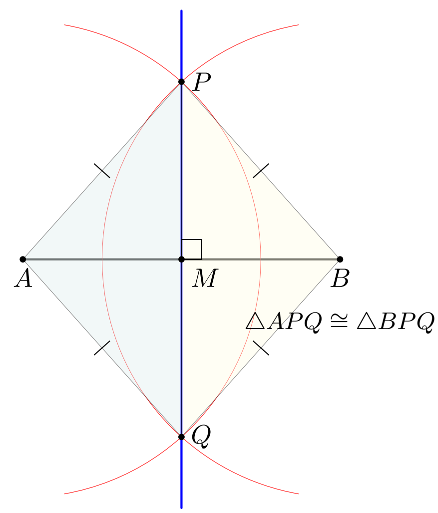
Therefore, in triangle APB, the segment PQ acts as an *angle bisector* from vertex P to side AB. Consequently, the point $M = PQ \cap AB$ is the point where this angle bisector meets the base.

But triangle APB is isosceles with $AP = BP$. In an isosceles triangle, the angle bisector from the top vertex is also a median and an altitude. Therefore:

$$AM = MB, \qquad \text{and} \qquad PQ \perp AB.$$

Hence the constructed line PQ is indeed the *perpendicular bisector* of segment AB, and the construction is correct.





Drawing a Perpendicular Through a Point Outside a Line

Problem. Given a line ℓ and a point P not on the line, construct a line through P that is perpendicular to ℓ.

Construction.

1. With center P and any radius large enough to reach ℓ, draw a circle. Let it intersect ℓ at two points; call them A and B.
2. With the same radius $PA = PB$, draw a circle centered at A and another circle with the same radius centered at B.
3. Let the two new circles intersect at a point Q on the opposite side of ℓ from P.
4. Draw the line PQ. This is the required perpendicular to ℓ.

Why it works (reasoning). Points A and B lie on the first circle centered at P, so

$$PA = PB.$$

Thus triangle PAB is isosceles.

Points Q and P lie on both circles centered at A and B, so

$$AQ = AP, \qquad BQ = BP.$$

Hence triangles PAQ and PBQ are congruent by **SSS**.

Therefore,

$$\angle APQ = \angle QPB,$$

so PQ is the angle bisector of the vertex angle of the isosceles triangle APB.

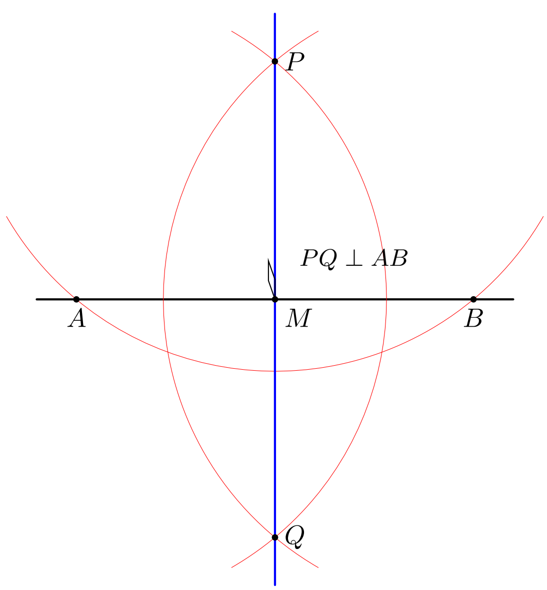
But in an isosceles triangle, the angle bisector from the top vertex is also the altitude. Thus the angle at the base is split into two right angles, so

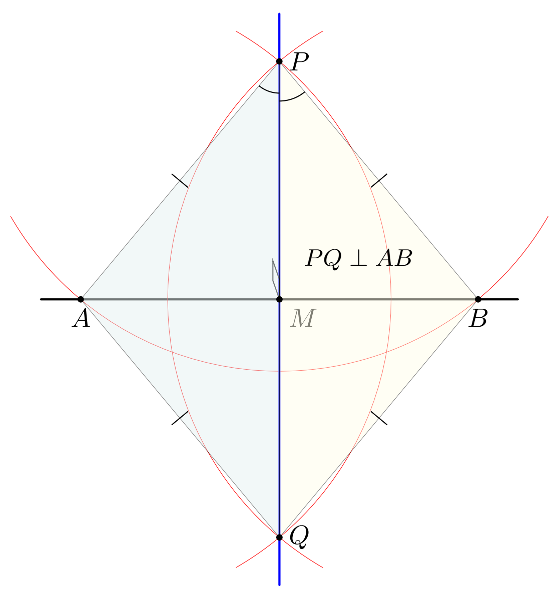
$$PQ \perp AB,$$

and since $A, B \in \ell$, we conclude

$$PQ \perp \ell.$$

The line PQ is therefore the perpendicular to ℓ through the point P.





Drawing a Perpendicular Through a Point on a Line

Problem. Given a line ℓ and a point P lying on the line, construct a line through P that is perpendicular to ℓ.

Construction.

1. Choose a radius r and draw a circle centered at P. It intersects ℓ at two points; call them A and B.
2. With a larger radius R, draw a circle centered at A and another circle centered at B.
3. Let these two circles intersect at point Q above the line.
4. Draw line PQ. This is the perpendicular to ℓ at P.

Why it works (reasoning). Points A and B lie on the first circle centered at P, so

$$PA = PB.$$

Point Q lies on both circles centered at A and B, so

$$AQ = BQ = r.$$

Triangle ABQ is isosceles, and point P is the middle of the base, therefore QP is a median.

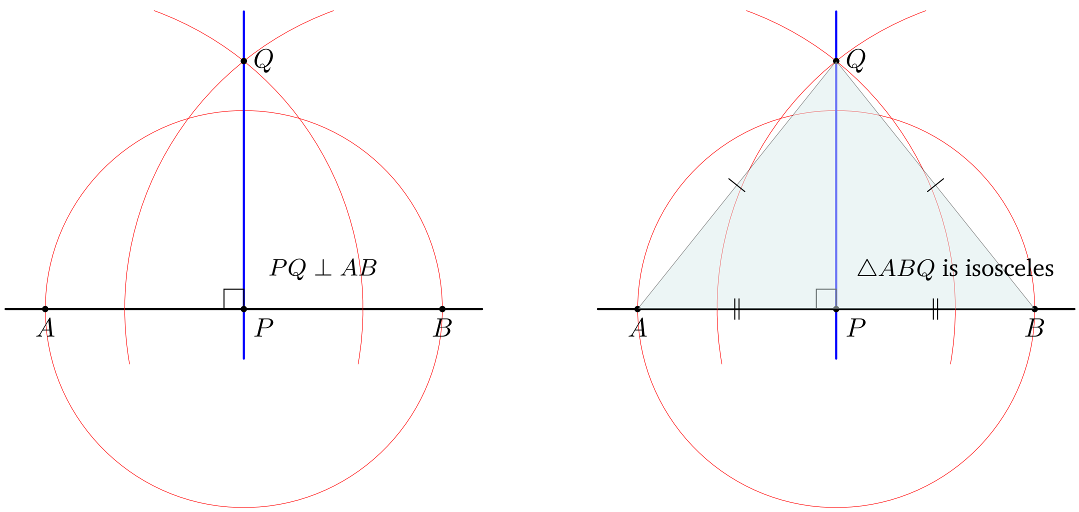
But in an isosceles triangle, the median from the top vertex (here, the vertex is Q) is also the altitude. Thus it is perpendicular to the base AB:

$$PQ \perp AB.$$

Since A and B lie on line ℓ, the base AB is part of ℓ, and therefore

$$PQ \perp \ell.$$

So the line through P perpendicular to ℓ has been correctly constructed.



Bisecting an Angle

Problem. Given an angle $\angle AOB$, construct its bisector: a ray from O that splits the angle into two equal angles.

Construction.

1. With center at O and any convenient radius, draw a circle that intersects the sides (rays) of the angle at points C on ray OA and D on ray OB.
2. With a radius greater than $\frac{1}{2}CD$, for example radius CD, draw a circle centered at C and a circle centered at D.
3. Let these two circles intersect at a point E inside the angle.
4. Draw the ray OE. This ray is the *angle bisector* of $\angle AOB$.

Why it works (reasoning). Points C and D lie on the same circle centered at O, so

$$OC = OD.$$

Points E lies on both circles centered at C and D, so

$$CE = DE.$$

Also, OE is a common side in triangles $\triangle OCE$ and $\triangle ODE$. Thus we have three pairs of equal sides:

$$OC = OD, \quad CE = DE, \quad OE = OE.$$

Therefore

$$\triangle OCE \cong \triangle ODE \quad \text{(by **SSS**)}.$$

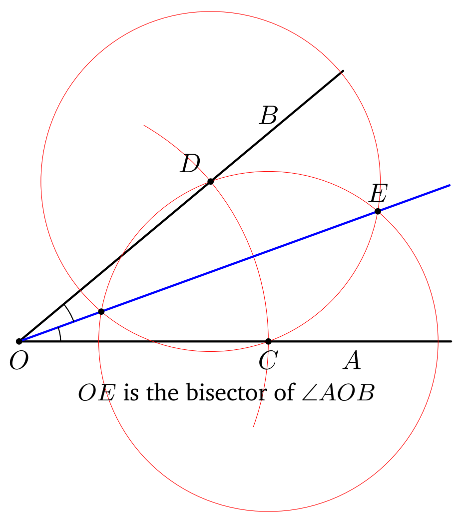
From the congruence, the angles at O in these triangles are equal:

$$\angle COE = \angle EOD.$$

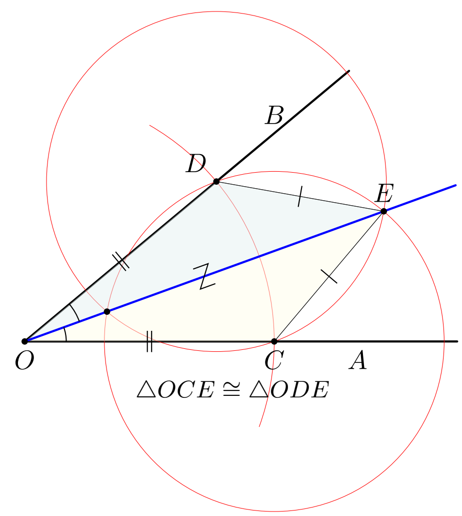
Therefore the ray OE bisects the angle:

$$\angle COE = \angle EOD = \tfrac{1}{2}\angle AOB,$$

and the construction is correct.



OE is the bisector of $\angle AOB$



$\triangle OCE \cong \triangle ODE$

Constructing a Regular Hexagon

Problem. Using a compass and ruler, construct a regular hexagon.

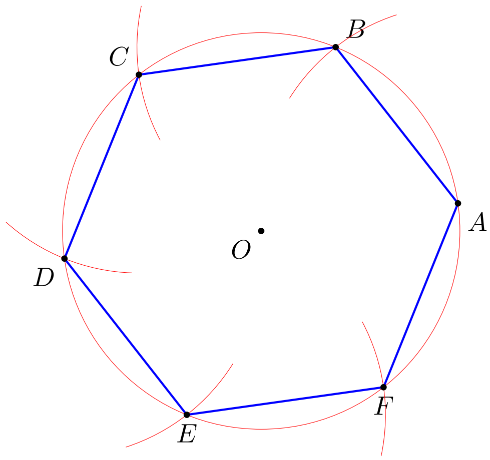
Construction (inscribed in a circle).

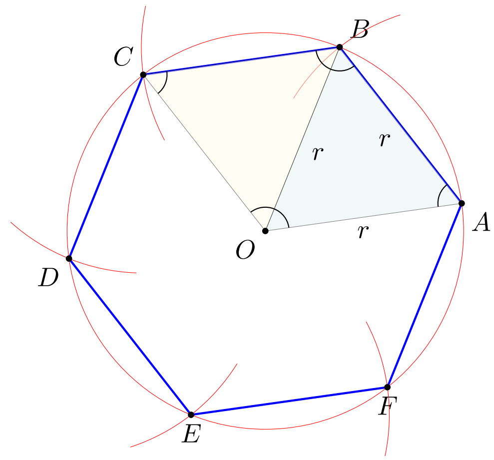
1. Draw a circle with center O and radius r. Choose a point A on the circle.
2. Without changing the compass width (still r), step off successive points on the circle: draw a circle centered at A of radius r to intersect the big circle at B; then centered at B to get C; continue to get D, E, F; connect $A - B - C - D - E - F - A$.

Why it works (reasoning). Triangles $\triangle AOB, \triangle BOC$, etc. are equilateral by construction. Therefore, each central angle $\angle AOB, \angle BOC$, etc. is equal to $60°$. This means that stepping the same chord six times yields six equal central angles that sum to $360°$, as we need.

Also, all angle of the resulting hexagon $\angle FAB, \angle ABC$, etc. are equal, since they are all $120°$.

Hence the constructed polygon has *equal sides and equal angles* — it is a regular hexagon.





Summary: Straightedge–and–Compass Constructions

In this section we learned how to build many basic geometric objects using only a straightedge and a compass. Almost every construction relied on the same core ideas:

- copying a length with a compass,
- intersecting circles and lines,
- creating isosceles (or equilateral) triangles,
- and using triangle congruence (especially SSS) to justify each step.

Here are the main constructions we developed:

- **Equilateral triangle.** From a segment AB, two circles of radius AB (centered at A and B) intersect at C, giving $\triangle ABC$ with $AB = AC = BC$.
- **Triangle from three sides (SSS).** Given lengths a, b, c, we construct a triangle with these sides *iff* the triangle inequality holds: the longest side is strictly shorter than the sum of the other two (for example, if a is longest, then $a < b + c$).
- **Copying an angle.** By reproducing the same arc and chord lengths from a new vertex, we construct an angle congruent to a given one, using SSS congruence of the two "arc triangles".
- **Bisecting a segment (perpendicular bisector).** Two equal circles centered at the endpoints of AB intersect; their line of centers is the perpendicular bisector, giving the midpoint M with $AM = MB$ and $PQ \perp AB$.
- **Perpendicular through a point (on or off the line).** Using circles to create isosceles triangles, we produced a line through a given point that is perpendicular to a given line – both for a point on the line and a point outside it.
- **Bisecting an angle.** Equal circles from the sides of $\angle AOB$ intersect at a point whose connection to the vertex gives a ray that splits the angle into two equal parts.
- **Regular hexagon in a circle.** Stepping a chord equal to the radius around a circle six times divides the circle into six equal central angles of $60°$, producing a regular hexagon with all sides and angles equal.

Together, these constructions form a basic "toolkit" of classical geometry. More complicated figures—such as other regular polygons and intricate geometric designs—are built by combining these fundamental moves.

Quick Check

4. What congruence test is most commonly used to justify compass-and-straightedge constructions?
5. When bisecting a segment AB, you construct two circles. Where must the centers of these circles be?
6. Each interior angle of a regular hexagon measures how many degrees?

Key Takeaways

- Compass-and-straightedge constructions use only two tools: a straightedge (to draw lines) and a compass (to draw circles and copy lengths).
- The triangle inequality states that a triangle with sides a, b, c exists if and only if every side is less than the sum of the other two.
- The perpendicular bisector of a segment passes through all points equidistant from the endpoints.
- Triangle congruence (especially SSS) is the key tool for proving constructions are correct.
- A regular hexagon can be inscribed in a circle by stepping off chords equal to the radius.

Common Mistakes

- **Using measurements on the straightedge.** A true straightedge has no markings. You can only draw lines through two points—you cannot measure distances directly.
- **Forgetting to check the triangle inequality.** Before constructing a triangle from three sides, always verify that the longest side is less than the sum of the other two.
- **Drawing arcs too small.** When bisecting a segment, the circles must be large enough to intersect (radius greater than half the segment).
- **Confusing angle bisector and perpendicular bisector.** An angle bisector divides an *angle* into two equal parts; a perpendicular bisector divides a *segment* into two equal parts and is perpendicular to it.

Classwork

1. Determine which sets of lengths can form a triangle (use the triangle inequality):
   (a) 5, 7, 10
   (b) 3, 4, 8
   (c) 6, 6, 6
   (d) 2, 3, 5
2. Describe (in words) how to construct the perpendicular bisector of a segment AB.
3. Describe how to construct a $60°$ angle using only compass and straightedge.
   Hint: Think about equilateral triangles.
4. True or False:
   (a) The perpendicular bisector of a segment passes through the midpoint of the segment.
   (b) To bisect an angle, you need to know its exact measurement in degrees.
   (c) A regular hexagon has all sides equal and all angles equal.
5. In a regular hexagon inscribed in a circle of radius r:
   (a) What is the length of each side?
   (b) What is the measure of each interior angle?

Classwork Solutions

1. (a) Yes: $5 + 7 = 12 > 10$, $5 + 10 = 15 > 7$, $7 + 10 = 17 > 5$.
   (b) No: $3 + 4 = 7 < 8$.
   (c) Yes: $6 + 6 = 12 > 6$ (equilateral triangle).
   (d) No: $2 + 3 = 5$, not strictly greater than 5.
2. Draw circles of the same radius (greater than half of AB) centered at A and B. These circles intersect at two points P and Q. Draw line PQ. This line is the perpendicular bisector, and it crosses AB at the midpoint M.
3. Construct an equilateral triangle on any segment AB. Since all angles of an equilateral triangle are $60°$, each angle at A and B is a $60°$ angle.
4. (a) True — that's the definition of a bisector.
   (b) False — the construction works without knowing the angle measure.
   (c) True — that's the definition of "regular."
5. (a) Each side equals the radius r (since each triangle OAB, OBC, etc. is equilateral).
   (b) Each interior angle is $120°$ (sum of angles is $(6 - 2) \times 180° = 720°$, divided by 6).

Homework

1. Determine which sets of lengths can form a triangle:

   (a) $4, 5, 6$

   (b) $1, 1, 3$

   (c) $7, 7, 7$

   (d) $8, 15, 17$

2. Describe how to construct an angle bisector for angle $\angle AOB$.

3. M You want to construct a $30°$ angle. Describe how to do this using the constructions you know. *Hint: Start with a $60°$ angle.*

4. M Explain why the perpendicular bisector construction works. Which triangle congruence test is used?

5. A segment PQ has length 8 cm. You want to find its midpoint using compass and straightedge.

   (a) What is the minimum radius you should use for your compass?

   (b) What is the length PM after the construction?

6. M Using compass and straightedge, how would you construct a square? Describe the steps.

7. H Prove that the angle bisector construction is correct. That is, show that if you follow the construction steps, the resulting ray divides the angle into two equal parts.

8. M A regular octagon (8 sides) is inscribed in a circle. What is the measure of each central angle? What is the measure of each interior angle of the octagon?

9. M Explain how to construct a rectangle with given side lengths a and b using only compass and straightedge (no protractor!). *Hint: You'll need to construct perpendicular lines.*

10. M Using compass and straightedge, construct an equilateral triangle, then bisect one of its angles. What angle have you constructed?

Quick Check Answers

1. Yes, because $3 + 4 = 7 > 5$, $3 + 5 = 8 > 4$, $4 + 5 = 9 > 3$. (This is also a right triangle!)
2. No, because $1 + 2 = 3 < 5$. The triangle inequality fails.
3. All three angles are $60°$ (since $180° \div 3 = 60°$).
4. SSS (Side-Side-Side) congruence.
5. At points A and B (the endpoints of the segment).
6. $120°$ (since $(6 - 2) \times 180° = 720°$, and $720° \div 6 = 120°$).