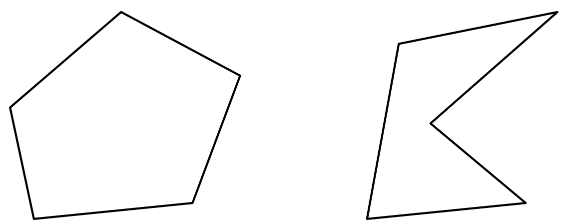
MATH 5: HANDOUT 17

GEOMETRY IV: POLYGONS AND QUADRILATERALS

Polygons and Quadrilaterals

A **polygon** is a closed figure formed by straight segments (called *sides*) joined end to end. Polygons are named by the number of sides: triangle (3), quadrilateral (4), pentagon (5), and so on.

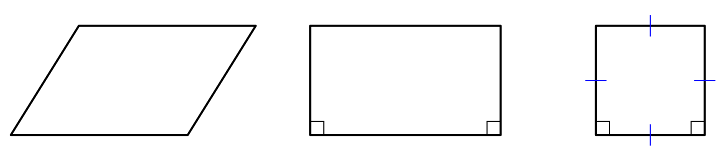
Convex vs. Concave. A polygon is **convex** if every interior angle is less than $180°$ (equivalently, any segment joining two interior points stays inside). If a polygon has a "dent" (some interior angle greater than $180°$), it is **concave**.



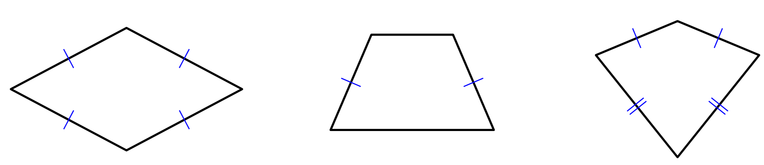
convex concave

Special Quadrilaterals (4-sided polygons).

- **Parallelogram:** both pairs of opposite sides are parallel.
- **Rectangle:** a parallelogram with four right angles.
- **Rhombus:** a parallelogram with four equal sides.
- **Square:** a rectangle and a rhombus (four equal sides and four right angles).
- **Trapezoid (US):** at least one pair of opposite sides parallel. *Isosceles trapezoid*: non-parallel sides equal, base angles equal.
- **Kite:** two adjacent pairs of equal sides.



parallelogram rectangle square



rhombus isosceles trapezoid kite

How quadrilaterals are related. Many quadrilaterals are *special cases* of others. For example, a square is not only a square — it is also a rectangle, a rhombus, and a parallelogram!

Hierarchy of quadrilaterals.

- **All squares** are rectangles, rhombi, parallelograms, and quadrilaterals.
- **All rectangles** are parallelograms (but not all parallelograms are rectangles).
- **All rhombi** are parallelograms (but not all parallelograms are rhombi).
- **All parallelograms** are quadrilaterals (but not all quadrilaterals are parallelograms).
- **Trapezoids** and **kites** are also quadrilaterals, but belong to different families.

Question: Which of the quadrilaterals are also **kites**?

Sum of Angles in a Quadrilateral

We already know that in any triangle, the sum of the three interior angles is

$$\angle A + \angle B + \angle C = 180^\circ.$$

What about four–sided figures (quadrilaterals)? It turns out that *any* quadrilateral—no matter how it looks—always has its four interior angles adding up to 360°.

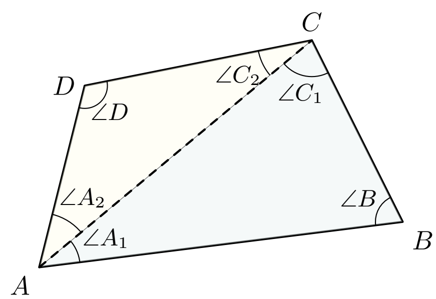
Theorem

Theorem. In any quadrilateral, the sum of the four interior angles is

$$\angle A + \angle B + \angle C + \angle D = 360^\circ.$$

Proof. Consider a quadrilateral $ABCD$. Draw a diagonal AC so that the quadrilateral is split into two triangles:

$$\triangle ABC \quad \text{and} \quad \triangle ACD.$$



By the triangle angle–sum theorem, we know:

$$\angle A_1 + \angle B + \angle C_1 = 180^\circ \quad \text{(in triangle } ABC\text{)},$$

$$\angle A_2 + \angle C_2 + \angle D = 180^\circ \quad \text{(in triangle } ACD\text{)}.$$

Here, $\angle C_1$ and $\angle C_2$ are the two angles at vertex C that lie inside the two triangles, and $\angle A_1$ and $\angle A_2$ are the two angles at vertex A inside the two triangles.

$$\angle A_1 + \angle A_2 = \angle A \quad \text{(together they make the interior angle at } A.\text{)}$$

$$\angle C_1 + \angle C_2 = \angle C \quad \text{(together they make the interior angle at } C.\text{)}$$

Add the two triangle equations:

$$(\angle A_1 + \angle B + \angle C_1) + (\angle A_2 + \angle C_2 + \angle D) = 180^\circ + 180^\circ = 360^\circ.$$

Group the angles:

$$(\angle A_1 + \angle A_2) + \angle B + (\angle C_1 + \angle C_2) + \angle D = 360^\circ.$$

Using $\angle A_1 + \angle A_2 = \angle A$ and $\angle C_1 + \angle C_2 = \angle C$, this becomes

$$\angle A + \angle B + \angle C + \angle D = 360^\circ.$$

Thus, in any quadrilateral, the sum of the four interior angles is 360°. □

Check it on familiar shapes

- In a rectangle or a square, each angle is 90°, so $90^\circ + 90^\circ + 90^\circ + 90^\circ = 360^\circ$.
- In a general parallelogram, opposite angles are equal and adjacent angles are supplementary. For example, if one angle is 110°, the next is 70°, so the four angles are $110^\circ, 70^\circ, 110^\circ, 70^\circ$, and they still sum to 360°.

In every case, however distorted the quadrilateral looks, its four angles always add up to 360°.

Sum of Angles in Polygons

The idea we used for quadrilaterals works for *any* polygon: break the shape into triangles by drawing diagonals from one vertex.

For an n-sided polygon ($n \geq 3$), we can draw $n - 3$ diagonals from a single vertex, splitting the polygon into $n - 2$ triangles.
Since each triangle has angle sum 180°, the total interior angle sum is

$$\boxed{\text{Sum of angles} = (n-2) \cdot 180^\circ.}$$

Examples.

- For a quadrilateral ($n = 4$): $(4 - 2) \cdot 180^\circ = 2 \cdot 180^\circ = 360^\circ$.
- For a pentagon ($n = 5$): $(5 - 2) \cdot 180^\circ = 540^\circ$.
- For a hexagon ($n = 6$): $(6 - 2) \cdot 180^\circ = 720^\circ$.

This works for any polygon, and the idea comes down to one simple fact: *every polygon is made of triangles.*

Quick Check

1. Three angles of a quadrilateral are 85°, 90°, and 110°. Find the fourth angle.
2. What is the sum of the interior angles of a heptagon (7-sided polygon)?
3. In a regular hexagon, all angles are equal. What is each angle?

Main Properties of Quadrilaterals

Each special type of quadrilateral has its own set of characteristic properties. Some of these properties define the shape itself (for example, a rectangle has right angles), and others describe interesting relationships, such as equal diagonals or perpendicular diagonals.

Let's review the main properties of the most common quadrilaterals.

Parallelogram. Opposite sides are parallel and equal. Opposite angles are also equal, and the diagonals bisect each other. Each diagonal divides the parallelogram into two congruent triangles. We will prove some of these properties, and leave the rest for later.

Theorem

Theorem 1. In a parallelogram, opposite sides are equal in length.

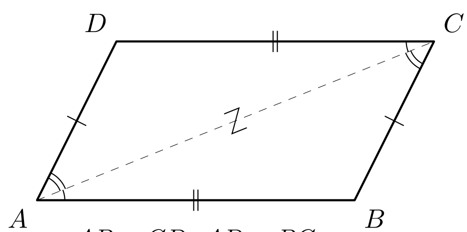
Proof. Let $ABCD$ be a parallelogram with $AB \parallel CD$ and $AD \parallel BC$.

Draw diagonal AC. Triangles ABC and CDA share the side AC. Because $AB \parallel CD$, alternate interior angles $\angle BAC$ and $\angle ACD$ are equal. Because $AD \parallel BC$, alternate interior angles $\angle CAD$ and $\angle ACB$ are equal.

Thus, the two triangles have:

$$\angle CAD = \angle ACB, \quad \angle BAC = \angle ACD, \quad AC \text{ in common.}$$

By the **ASA** congruence test, $\triangle ABC \cong \triangle CDA$. Therefore, $AB = CD$ and $AD = BC$. □



$AB = CD,\ AD = BC$

Theorem

Theorem 2 (Converse). If in a quadrilateral both pairs of opposite sides are equal, then the quadrilateral is a parallelogram.

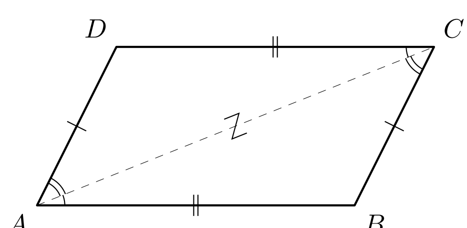
Proof. Let $ABCD$ be a quadrilateral with $AB = CD$ and $AD = BC$.

Draw diagonal AC. In triangles ABC and CDA:

$$AB = CD, \quad AD = BC, \quad AC \text{ is common.}$$

By the **SSS** congruence test, $\triangle ABC \cong \triangle CDA$.

From the congruence, corresponding angles are equal: $\angle CAB = \angle ACD$ and $\angle CAD = \angle ACB$. These pairs of equal alternate interior angles show that $AB \parallel CD$ and $AD \parallel BC$. Hence $ABCD$ is a parallelogram. □



$AB \parallel CD,\ AD \parallel BC$

Note

Because of this converse theorem, we can actually *define* a parallelogram either as a quadrilateral with both pairs of opposite sides parallel, *or* as a quadrilateral with both pairs of opposite sides equal. These two definitions are equivalent in Euclidean geometry.

Rectangle. A rectangle is a parallelogram with all angles equal to $90°$. Because of its symmetry, its diagonals are equal in length and bisect each other.

Rhombus. A rhombus is a parallelogram with all sides equal. Its diagonals are perpendicular and bisect each other, dividing the rhombus into four congruent right triangles. Opposite angles are equal.

Square. A square has all the properties of both a rectangle and a rhombus:

- all sides equal,
- all angles right,
- diagonals equal, perpendicular, and bisect each other.

The square is the most symmetric of all quadrilaterals.

Trapezoid. A trapezoid has at least one pair of opposite sides parallel, called *bases*. The height is the perpendicular distance between the bases. In an **isosceles trapezoid**, the non-parallel sides are equal and the base angles are equal. Its diagonals are also equal in length.

Kite. A kite has two pairs of adjacent equal sides. One diagonal is an axis of symmetry: it bisects the other diagonal and the angles at its ends. The diagonals are perpendicular.

All these properties are summarized below:

Summary of Main Properties.

Shape	Opp. sides $\parallel$	Opp. sides $=$	Adj. sides $=$	All sides $=$	Right angles	Diag. $=$	Diag. $\perp$
Parallelogram	✓	✓					
Rectangle	✓	✓			✓	✓	
Rhombus	✓	✓	✓	✓			✓
Square	✓	✓	✓	✓	✓	✓	✓
Trapezoid	(1 pair)						
Isos. Trapezoid	(1 pair)	(1 pair)				✓	
Kite			✓				✓

Summary

- All parallelograms share many common properties (equal and parallel opposite sides, bisecting diagonals).
- Rectangles and rhombi each add one more special condition (equal right angles or equal sides).
- The square combines both sets of properties — it is both a rectangle and a rhombus.
- Trapezoids and kites belong to different families but also show useful symmetries.

Quick Check

4. True or False: Every square is a rhombus.
5. True or False: Every parallelogram is a rectangle.
6. A parallelogram has one angle measuring 65°. Find the other three angles.

Areas of Quadrilaterals

Area measures how much surface a figure covers. We already know the area of a triangle is $\frac{1}{2}bh$. Now we can find the areas of the most common quadrilaterals.

Square. A square has all sides equal and all angles right. If the side of a square is a, then by definition of area units:

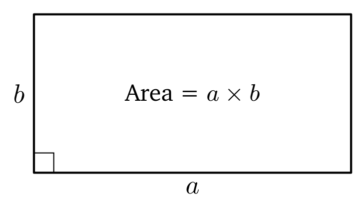
$$\text{Area of a square} = a^2$$

This rule is actually used as the *definition* of area for all rectangles: the number of unit squares that fit along the sides (even if those sides are not whole numbers).

Rectangle. A rectangle has opposite sides equal and all angles right. If its sides are a and b:

$$\text{Area of a rectangle} = a \times b$$

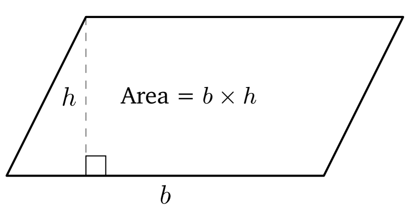
This formula works for any lengths a and b. It agrees with the idea that if a or b are not whole numbers, area can still be found by multiplication rather than by counting unit squares.

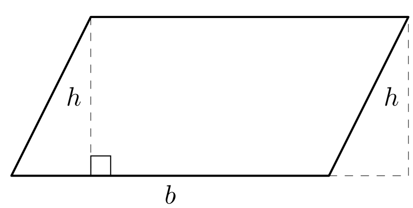


Parallelogram. A parallelogram has opposite sides parallel and equal. Let the base be b and the height (the perpendicular distance between opposite sides) be h.

$$\text{Area of a parallelogram} = b \times h$$

Proof. Cut a right triangle from one side of the parallelogram and move it to the other side. The new figure is a rectangle with the same base and height. Since cutting and rearranging do not change area, the parallelogram and rectangle have equal areas, so the area is $b \times h$. □

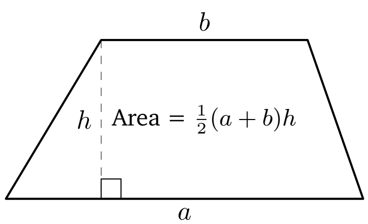


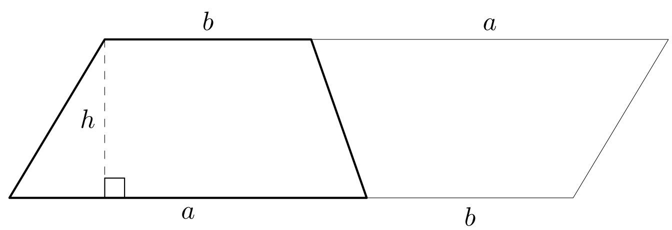


Trapezoid. A trapezoid has one pair of parallel sides, called the *bases*. Let the lengths of the bases be a and b, and let h be the distance (height) between them.

$$\text{Area of a trapezoid} = \frac{(a+b)}{2} \times h$$

Proof. Take two identical trapezoids and join them along a non-parallel side. They form a parallelogram whose base is $a+b$ and height is h. The area of that parallelogram is $(a+b)h$. Since each trapezoid is half of it, its area is $\frac{1}{2}(a+b)h$. □





Summary of area formulas:

Square	:	a^2
Rectangle	:	$a \times b$
Parallelogram	:	$b \times h$
Trapezoid	:	$\frac{1}{2}(a+b) \times h$

Quick Check

7. Find the area of a parallelogram with base 12 and height 5.
8. Find the area of a trapezoid with bases 8 and 14 and height 6.

Key Takeaways

- The sum of interior angles in an n-sided polygon is $(n-2) \cdot 180°$.
- In a quadrilateral, the four angles sum to $360°$.
- A parallelogram has opposite sides equal and parallel; its diagonals bisect each other.
- A rectangle is a parallelogram with right angles; a rhombus has all sides equal; a square is both.
- Area formulas: rectangle $= a \times b$; parallelogram $= b \times h$; trapezoid $= \frac{1}{2}(a+b) \times h$.

Common Mistakes

- **Confusing height with side length.** In a parallelogram, the height is the *perpendicular* distance between the parallel sides, not the length of a slanted side.
- **Forgetting the hierarchy.** A square is also a rectangle, a rhombus, and a parallelogram. When asked ”Is every square a parallelogram?” the answer is yes!
- **Using the wrong angle sum.** For polygons with n sides, use $(n-2) \cdot 180°$, not $n \cdot 180°$.
- **Mixing up bases in trapezoid formula.** Both parallel sides are ”bases” — add them together, then multiply by half the height.

Classwork

1. Find the sum of the interior angles of:
   (a) a pentagon
   (b) an octagon (8 sides)
   (c) a decagon (10 sides)
2. In quadrilateral $ABCD$, $\angle A = 95°$, $\angle B = 80°$, and $\angle C = 100°$. Find $\angle D$.
3. A regular polygon has all angles equal to $140°$. How many sides does it have?
4. In parallelogram $PQRS$, $\angle P = 72°$. Find all four angles.
5. Classify each statement as True or False:
   (a) Every rectangle is a parallelogram.
   (b) Every rhombus is a square.
   (c) Every square is a rhombus.
   (d) A kite is a parallelogram.
6. Find the area of:
   (a) A rectangle with length 15 cm and width 8 cm
   (b) A parallelogram with base 12 m and height 7 m
   (c) A trapezoid with bases 10 and 16 and height 9
7. A parallelogram has area 84 square units and base 12. What is its height?

Classwork Solutions

1. **Sum of interior angles.**
   (a) Pentagon: $(5-2) \cdot 180° = 3 \cdot 180° = 540°$
   (b) Octagon: $(8-2) \cdot 180° = 6 \cdot 180° = 1080°$
   (c) Decagon: $(10-2) \cdot 180° = 8 \cdot 180° = 1440°$
2. $\angle D = 360° - 95° - 80° - 100° = 85°$.
3. Let the polygon have n sides. Each angle is $\frac{(n-2) \cdot 180°}{n} = 140°$.
   $(n-2) \cdot 180 = 140n$
   $180n - 360 = 140n$
   $40n = 360$
   $n = 9$ sides.
4. In a parallelogram, opposite angles are equal and adjacent angles are supplementary.
   $\angle P = \angle R = 72°$
   $\angle Q = \angle S = 180° - 72° = 108°$
5. (a) True — a rectangle has both pairs of opposite sides parallel.
   (b) False — a rhombus doesn't necessarily have right angles.
   (c) True — a square has all four sides equal.
   (d) False — a kite doesn't have parallel opposite sides.
6. (a) Rectangle: $15 \times 8 = 120$ cm²

(b) Parallelogram: $12 \times 7 = 84$ m²

(c) Trapezoid: $\frac{1}{2}(10 + 16) \times 9 = \frac{1}{2} \times 26 \times 9 = 117$ square units

7. $84 = 12 \times h \Rightarrow h = 7$ units.

Homework

1. Find the sum of the interior angles of a 12-sided polygon.
2. In quadrilateral $WXYZ$, three angles measure $88°$, $92°$, and $115°$. Find the fourth angle.
3. M Each interior angle of a regular polygon measures $156°$. How many sides does the polygon have?
4. In parallelogram $ABCD$, $\angle A = 3x + 10$ and $\angle B = 2x + 20$. Find the value of x and all four angles.
5. Find the area of a parallelogram with base 18 and height 11.
6. Find the area of a trapezoid with parallel sides 9 and 15 and height 8.
7. M A rectangle has perimeter 48 cm and length twice its width. Find its area.
8. M The diagonals of a rhombus are 10 and 24. Find the area of the rhombus.
   Hint: The diagonals divide the rhombus into four congruent right triangles.
9. M A trapezoid has area 120 square units. If its parallel sides are 10 and 14, find its height.
10. H Prove that the diagonals of a parallelogram bisect each other.
    Hint: Use triangle congruence.
11. M In a regular hexagon (6-sided polygon with all sides and angles equal):
    (a) Show that each interior angle is $120°$.
    (b) Show that opposite sides are parallel.
    Hint: Show that alternate interior angles are equal to $60°$.
12. M Can you cut a trapezoid into pieces from which you can construct a rectangle? Explain how.
13. M In rectangle $ABCD$, let M be the midpoint of side BC. Prove that triangle AMD is isosceles (i.e., $AM = DM$).
    Hint: Use the fact that opposite sides of a rectangle are equal.
14. H Prove: If in a quadrilateral $ABCD$, the opposite angles are equal ($\angle A = \angle C$ and $\angle B = \angle D$), then $ABCD$ is a parallelogram.
    Hint: Use the fact that the sum of all angles in a quadrilateral is $360°$.

Quick Check Answers

1. $360° - 85° - 90° - 110° = 75°$
2. $(7-2) \cdot 180° = 5 \cdot 180° = 900°$
3. Sum $= (6-2) \cdot 180° = 720°$, so each angle $= 720° \div 6 = 120°$
4. True — a square has all four sides equal, which is the definition of a rhombus.
5. False — a parallelogram doesn't necessarily have right angles.
6. Opposite angles equal: $65°$ and $65°$. Adjacent angles supplementary: $115°$ and $115°$.
7. Area $= 12 \times 5 = 60$ square units.
8. Area $= \frac{1}{2}(8 + 14) \times 6 = \frac{1}{2} \times 22 \times 6 = 66$ square units.