

MATH 5: HANDOUT 16

GEOMETRY III: AREA AND CONGRUENCE

Area

Before we find the area of a triangle, we must understand what **area** is in general. Surprisingly, giving a completely precise definition of area for *all* possible shapes (especially shapes with curved or complicated boundaries) is very difficult and belongs to advanced mathematics (real analysis and measure theory). In this course we will not attempt the full formal definition. Instead, we will describe the basic ideas and the properties that area must satisfy.

What is area? The *area* of a plane figure is a number that describes how much *space* the figure occupies. To define area, we need a standard unit.

Unit square. A *unit square* is a 1×1 square. By definition, its area is

$$\text{Area of unit square} = 1.$$

All other areas are measured by comparing shapes to the unit square.

Basic properties of area. Any reasonable concept of area (for polygons and simple figures) must satisfy the following properties:

1. **Congruence property.** If two figures are congruent (one can be moved or rotated to match the other exactly), then they have the same area.
2. **Additivity.** If a figure is cut into non-overlapping pieces, then

$$\text{Area of the whole} = \text{sum of the areas of the pieces.}$$

3. **Monotonicity.** If one figure is entirely contained inside another, then its area is smaller or equal.
4. **Normalization.** The unit square has area 1.

Using only these basic rules, we can already compute the areas of many shapes, especially polygons such as rectangles and triangles.

Rectangles. A $w \times h$ rectangle can be divided into w columns and h rows of unit squares (assuming w and h are whole numbers). By additivity,

$$\text{Area of rectangle} = w \cdot h.$$

This formula also remains true for rectangles with fractional side lengths.

Why defining area precisely is hard

For shapes with curved or very irregular boundaries (for example, a region bounded by a wiggly curve), the simple methods above are not enough. In advanced mathematics, the area of such shapes is defined using limits of smaller and smaller partitions — this leads to the theory of *measure*, an important subject in university-level mathematics.

In this course, we will work only with polygons and simple curved shapes whose areas can be found using basic methods.

Area of a Triangle

Theorem

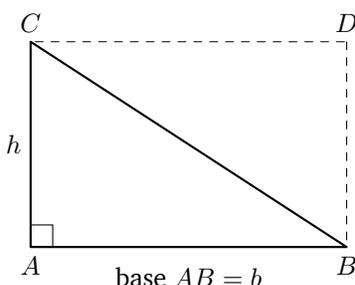
Formula. The area of a triangle equals

$$\text{Area} = \frac{1}{2} b \cdot h,$$

where b is the length of a base and h is the corresponding height (altitude).

Proof. Let ABC be any triangle and take AB as the base with length b . Drop the altitude from C to the (possibly extended) line AB ; let the foot be H and the altitude length be h (so $CH \perp AB$).

Step 1 (Right triangles have area $\frac{1}{2}bh$). If a triangle is right with legs b and h , it fits exactly into a rectangle of sides b and h as one of two equal halves. Thus a right triangle's area is $\frac{1}{2}bh$.



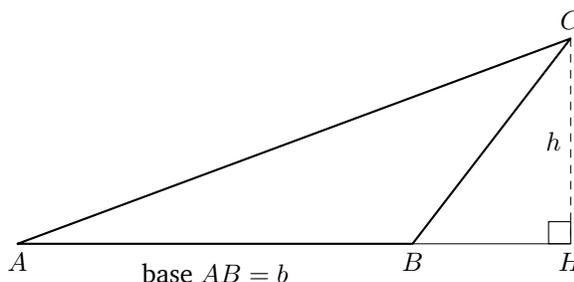
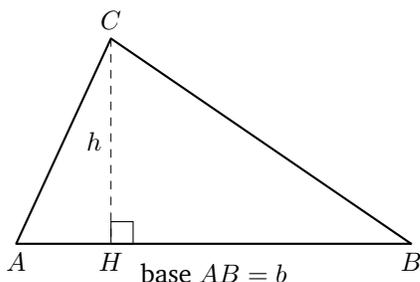
Step 2 (Acute case: H lies between A and B). Then ABC is the union of two right triangles: $\triangle ACH$ and $\triangle BCH$, which share the same height h and bases AH and HB . Hence

$$[ABC] = [ACH] + [BCH] = \frac{1}{2}(AH)h + \frac{1}{2}(HB)h = \frac{1}{2}(AH + HB)h = \frac{1}{2}bh.$$

Step 3 (Obtuse case: H lies outside segment AB). Assume H is beyond B (the other side is analogous). Then the larger right triangle $\triangle ACH$ contains $\triangle ABC$ and the right triangle $\triangle BCH$. Therefore

$$[ABC] = [ACH] - [BCH] = \frac{1}{2}(AH)h - \frac{1}{2}(HB)h = \frac{1}{2}(AH - HB)h = \frac{1}{2}ABh = \frac{1}{2}bh.$$

In all cases, $[ABC] = \frac{1}{2}bh$. □



Example 1. If $b = 10$ and $h = 3$, then $\text{Area} = \frac{1}{2} \cdot 10 \cdot 3 = 15$.

A triangle has area 30 square units and base 12. Find the height.

Solution: Using $\text{Area} = \frac{1}{2}bh$, we have $30 = \frac{1}{2} \cdot 12 \cdot h$, so $30 = 6h$, giving $h = 5$.

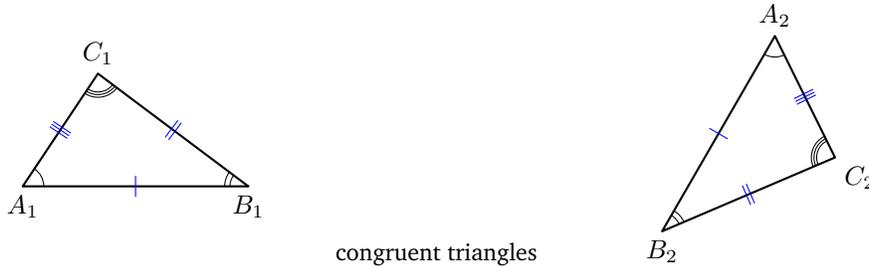
Example 3 (Obtuse triangle). In an obtuse triangle, the height to a side may fall *outside* the triangle. The formula $\text{Area} = \frac{1}{2}bh$ still works—we just need to use the perpendicular distance from the opposite vertex to the (extended) base line.

Quick Check

1. Find the area of a triangle with base 8 and height 5.
2. A triangle has area 24 and base 6. Find the height.

Congruence of Triangles

Two figures are **congruent** if one can be moved (by sliding, turning, or flipping) to fit exactly on top of the other. For triangles, “congruent” means all three sides and all three angles match pairwise.



Checking all six parts (three sides and three angles) every time is slow. Happily, in Euclidean geometry we can use three famous shortcuts (called *congruence tests*):

Theorem

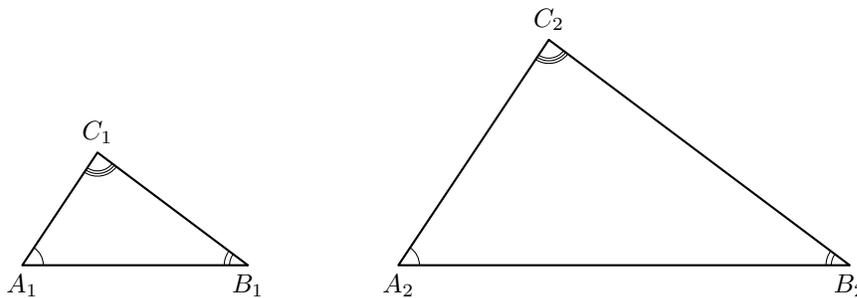
Triangle Congruence Tests.

1. **SSS** (side–side–side): If three sides of one triangle equal three sides of another, the triangles are congruent.
2. **SAS** (side–angle–side): If two sides and the angle between them of one triangle equal the corresponding parts of another, the triangles are congruent.
3. **ASA** (or **AAS**): If two angles and a side of one triangle equal the corresponding parts of another, the triangles are congruent.

Why other “tests” fail.

- **AAA** (three angles equal) *does not* guarantee congruence — only *similarity*. The triangles can be different sizes.

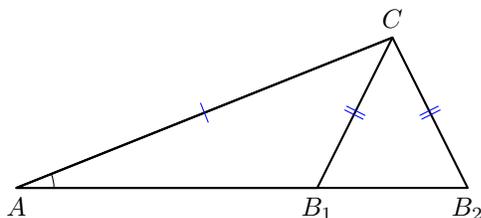
In the picture below, in triangles $A_1B_1C_1$ and $A_2B_2C_2$, $\angle B_1A_1C_1 = \angle B_2A_2C_2$, $\angle A_1B_1C_1 = \angle A_2B_2C_2$, and $\angle A_1C_1B_1 = \angle A_2C_2B_2$, but triangles differ in size (such triangles are called **similar**).



AAA: similar only

- **SSA** (two sides and a non-included angle) *does not* guarantee congruence — it is the *ambiguous case*. The same data can produce two different triangles (or sometimes none).

In the picture below, in triangles AB_1C and AB_2C , $\angle B_1AC = \angle B_2AC$, AC is a common side, and $B_1C = B_2C$, but these triangles are not congruent.



SSA: two non-congruent triangles

- **Exception: HL (Hypotenuse-Leg)**. There is one important case where SSA *does* work: in **right triangles**. If two right triangles have equal hypotenuses and one pair of equal legs, then they are congruent. This is called the **HL (Hypotenuse-Leg)** test.

Why does it work? The right angle eliminates the ambiguity. In the SSA picture above, the two possible positions for B arise because the circle can intersect the ray in two places. But if the angle at A is 90° , there is only one intersection point.

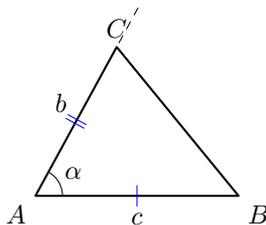
Why the congruence tests work (ruler-and-compass constructions). The congruence tests can be understood by asking: *if we try to construct a triangle with the given information using ruler and compass, is the result unique?* If the construction produces exactly one triangle (up to reflection), then any two triangles with the same given data must be congruent.

SAS (Side-Angle-Side). *Given:* Two sides b and c , and the included angle α .

Construction:

1. Draw segment AB of length c (using ruler).
2. At point A , construct angle α (using compass).
3. On the ray from A making angle α , mark point C at distance b (using compass).
4. Connect C to B (using ruler).

Why it's unique: Each step has exactly one outcome. Once we place A and B , the angle at A determines a unique ray, and the distance b determines a unique point C on that ray. The triangle is completely determined.



SAS: angle + two sides \Rightarrow unique triangle

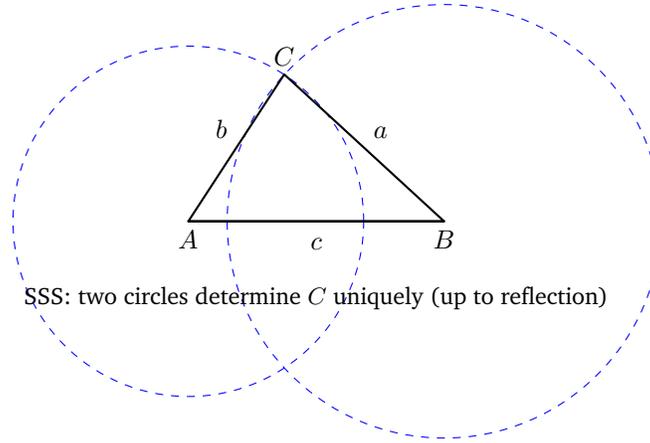
SSS (Side-Side-Side). *Given:* Three side lengths a , b , and c .

Construction:

1. Draw segment AB of length c (using ruler).
2. Draw a circle centered at A with radius b (using compass).

3. Draw a circle centered at B with radius a (using compass).
4. The circles intersect at point C . Connect A to C and B to C .

Why it's unique: Two circles can intersect in at most two points, and these two points are mirror images of each other across line AB . So there is exactly one triangle with sides a, b, c (up to reflection).



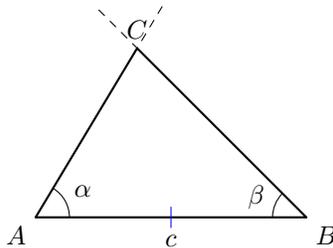
ASA (Angle-Side-Angle). *Given:* Two angles α and β , and the side c between them.

Construction:

1. Draw segment AB of length c (using ruler).
2. At point A , construct a ray making angle α with AB (using compass).
3. At point B , construct a ray making angle β with BA (using compass).
4. The two rays intersect at point C .

Why it's unique: Two non-parallel lines meet at exactly one point. The rays are not parallel because $\alpha + \beta < 180$ (the triangle must have a positive third angle). So C is uniquely determined.

AAS (Angle-Angle-Side). If we know two angles and a non-included side, we can compute the third angle (since angles sum to 180), which gives us ASA. So AAS also determines a unique triangle.



ASA: two rays at angles α and β meet at unique point C

Test	What you need	Result
SSS	3 sides equal	Congruent ✓
SAS	2 sides + included angle equal	Congruent ✓
ASA/AAS	2 angles + any side equal	Congruent ✓
HL	Right triangles: hypotenuse + leg equal	Congruent ✓
AAA	3 angles equal	Similar only ✗
SSA	2 sides + non-included angle	Ambiguous ✗

- Use **SSS**, **SAS**, **ASA/AAS**, or **HL** (for right triangles) to prove triangles are congruent.
- **AAA** gives only similarity; **SSA** is ambiguous (except for HL).
- Congruent triangles match exactly in size and shape.

Quick Check

3. In triangles ABC and DEF , we know $AB = DE$, $BC = EF$, and $\angle B = \angle E$. Are the triangles congruent? Which test applies?
4. Can two triangles with all three angles equal be different sizes? What is this relationship called?

Isosceles Triangles

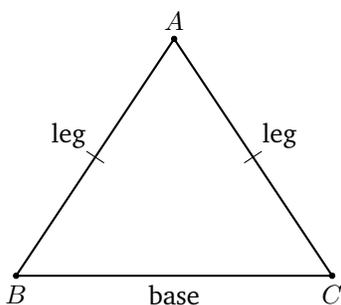
Not all triangles are created equal! Triangles with some symmetry often have many extra “hidden” properties. One of the most important examples is an **isosceles** triangle.

Definition. A triangle is called **isosceles** if it has two equal sides. The equal sides are called the **legs**, and the third side is called the **base**.

In our diagrams and problems we will usually name an isosceles triangle so that the legs are AB and AC , and the base is BC , with

$$AB = AC.$$

The vertex at the top (opposite the base) is A , and the *base angles* are the angles at B and C .



Isosceles triangles have a very special “top-to-base” segment: the segment from the top vertex A to the midpoint of the base BC turns out to be *all at once* a median, an angle bisector, and an altitude. This single fact explains most of the familiar properties of isosceles triangles.

We now state this as a theorem.

Theorem

Theorem (Properties of an isosceles triangle). Let ABC be a triangle with $AB = AC$, so BC is the base. Let M be the midpoint of BC , so that $BM = MC$.

Then:

1. The base angles are equal:

$$\angle B = \angle C.$$

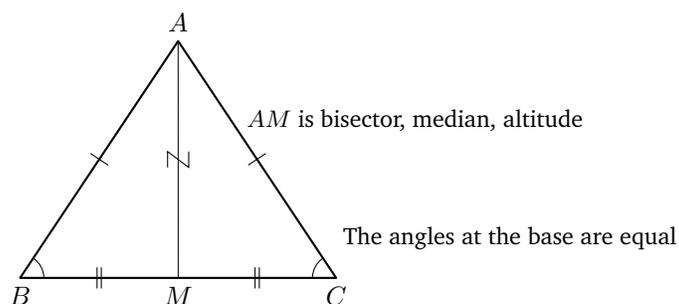
2. The segment AM is an angle bisector of $\angle A$:

$$\angle BAM = \angle MAC.$$

3. The segment AM is an altitude to the base BC :

$$AM \perp BC.$$

Proof. We will compare the two smaller triangles on the left and on the right of the segment AM .



Step 1. The two smaller triangles are congruent.

Consider triangles $\triangle ABM$ and $\triangle ACM$. We know:

- $AB = AC$ (given: ABC is isosceles),
- $BM = MC$ (because M is the midpoint of BC),
- $AM = AM$ (a common side).

Thus all three pairs of corresponding sides are equal, so

$$\triangle ABM \cong \triangle ACM \quad (\text{by SSS}).$$

Step 2. Base angles are equal (part 1).

From the congruence $\triangle ABM \cong \triangle ACM$, the angles at B and C are corresponding angles in the two triangles. Therefore

$$\angle B = \angle C.$$

This proves part (1): in an isosceles triangle, the base angles are equal.

Step 3. The median is an angle bisector (part 2).

The congruence also tells us that the two "top" angles at A in the small triangles are equal:

$$\angle BAM = \angle MAC.$$

So the segment AM splits $\angle A$ into two equal angles, which means that AM is an *angle bisector* of $\angle A$. This proves part (2).

Step 4. The median is also an altitude (part 3).

Finally, look at the angles at point M :

$$\angle BMA \text{ and } \angle CMA.$$

They are also corresponding angles in the congruent triangles, so they are equal:

$$\angle BMA = \angle CMA.$$

But the rays MB and MC lie on the same straight line BC , so the two adjacent angles at M add up to a straight angle:

$$\angle BMA + \angle CMA = 180^\circ.$$

If two adjacent angles are equal and sum to 180° , then each angle is 90° :

$$\angle BMA = \angle CMA = 90^\circ.$$

So AM is perpendicular to BC , i.e. $AM \perp BC$. This means AM is an *altitude* to the base BC . This proves part (3) and completes the proof. \square

Key Insight

The entire proof relies on one key observation: the two “half-triangles” $\triangle ABM$ and $\triangle ACM$ are congruent by **SSS**. Once we establish this congruence, all the other properties (equal base angles, angle bisector, altitude) follow immediately as corresponding parts of congruent triangles.

Summary

In an isosceles triangle ABC with $AB = AC$:

- the base angles are equal: $\angle B = \angle C$;
- the segment from the top vertex to the midpoint of the base (AM in our notation) is simultaneously
 - a median,
 - an angle bisector,
 - and an altitude.

Most isosceles-triangle problems secretly use this theorem, even if they do not mention it directly.

In short: In an isosceles triangle, geometry meets symmetry — one line (the median/altitude/bisector from the vertex) perfectly divides the figure into two matching halves.

The Converse: Equal Angles Imply Equal Sides

The converse of the isosceles triangle theorem is also true:

Theorem

Theorem (Converse). If a triangle has two equal angles, then it is isosceles (the sides opposite those angles are equal).

Proof. Suppose $\angle B = \angle C$ in triangle ABC . Draw the angle bisector from A to side BC , meeting it at point M .

In triangles ABM and ACM :

- $\angle ABM = \angle ACM$ (given: $\angle B = \angle C$)
- $\angle BAM = \angle CAM$ (since AM is the angle bisector)
- $AM = AM$ (common side)

By **AAS**, $\triangle ABM \cong \triangle ACM$, so $AB = AC$. Thus the triangle is isosceles. □

Equilateral Triangles

An **equilateral triangle** is a special case of an isosceles triangle where *all three sides* are equal:

$$AB = BC = CA.$$

Since an equilateral triangle is isosceles in three different ways (any side can be the “base”), we can apply the isosceles triangle theorem three times to conclude:

Theorem

In an equilateral triangle, all three angles are equal. Since the angles sum to 180° , each angle is exactly 60° .

The converse is also true: if all three angles of a triangle are equal (each 60°), then the triangle is equilateral.

Quick Check

5. In isosceles triangle ABC with $AB = AC$, if $\angle A = 40^\circ$, find $\angle B$ and $\angle C$.
6. In isosceles triangle PQR with $PQ = PR$, if $\angle Q = 65^\circ$, find $\angle P$.

Common Mistakes

- **Confusing height with a slant side.** The height of a triangle must be *perpendicular* to the base. In an obtuse triangle, the height may fall outside the triangle, but it’s still the perpendicular distance.
- **Mixing up SAS and SSA.** In SAS, the angle must be *between* the two sides (the “included angle”). If the angle is not between the sides, you have SSA, which does NOT guarantee congruence.
- **Thinking AAA proves congruence.** Three equal angles only prove the triangles are *similar* (same shape), not congruent (same size). You need at least one side to match.
- **Forgetting the converse.** “Equal sides \Rightarrow equal angles” is not the same as “equal angles \Rightarrow equal sides.” Fortunately, for isosceles triangles, both directions are true!
- **In isosceles triangles: confusing which angles are equal.** The equal angles are the *base angles* (opposite the equal sides), not the vertex angle.

Classwork

- Find the area of a triangle with:
 - base 12 cm and height 7 cm
 - base 15 m and height 8 m
 - base 9 in and height 4 in
- A triangle has area 36 square units. If its base is 9 units, what is its height?
- Two triangles share the same base of length 10. One has height 6 and the other has height 9. What is the ratio of their areas?
- For each pair of triangles, state whether they are congruent and which test (SSS, SAS, ASA, AAS) applies, or explain why they may not be congruent.
 - $\triangle ABC$: $AB = 5$, $BC = 7$, $CA = 8$
 $\triangle DEF$: $DE = 5$, $EF = 7$, $FD = 8$
 - $\triangle PQR$: $PQ = 6$, $\angle P = 50^\circ$, $PR = 8$
 $\triangle XYZ$: $XY = 6$, $\angle X = 50^\circ$, $XZ = 8$
 - $\triangle ABC$: $\angle A = 40^\circ$, $\angle B = 60^\circ$, $\angle C = 80^\circ$
 $\triangle DEF$: $\angle D = 40^\circ$, $\angle E = 60^\circ$, $\angle F = 80^\circ$
- In isosceles triangle ABC with $AB = AC = 10$ and base $BC = 12$:
 - Find the height from A to BC .
 - Find the area of the triangle.
- In isosceles triangle XYZ with $XY = XZ$, if $\angle Y = 55^\circ$, find all three angles.
- Prove that if a triangle has two equal angles, then it is isosceles.
Hint: This is the converse of the base angles theorem.

Classwork Solutions

1. Find the area of a triangle.

(a) $\text{Area} = \frac{1}{2} \cdot 12 \cdot 7 = 42 \text{ cm}^2$

(b) $\text{Area} = \frac{1}{2} \cdot 15 \cdot 8 = 60 \text{ m}^2$

(c) $\text{Area} = \frac{1}{2} \cdot 9 \cdot 4 = 18 \text{ in}^2$

2. Find the height.

$$36 = \frac{1}{2} \cdot 9 \cdot h \Rightarrow h = \frac{36 \cdot 2}{9} = 8 \text{ units.}$$

3. Ratio of areas.

$$\text{Area}_1 = \frac{1}{2} \cdot 10 \cdot 6 = 30, \text{Area}_2 = \frac{1}{2} \cdot 10 \cdot 9 = 45.$$

$$\text{Ratio} = 30 : 45 = 2 : 3.$$

4. Congruence tests.

(a) Yes, congruent by **SSS** (all three sides equal).

(b) Yes, congruent by **SAS** (two sides and the included angle equal).

(c) Not necessarily congruent — this is **AAA**, which only guarantees similarity. The triangles could be different sizes.

5. Isosceles triangle with $AB = AC = 10$, $BC = 12$.

(a) Let M be the midpoint of BC . Then $BM = 6$.

$$\text{By the Pythagorean theorem: } AM = \sqrt{AB^2 - BM^2} = \sqrt{100 - 36} = \sqrt{64} = 8.$$

(b) $\text{Area} = \frac{1}{2} \cdot 12 \cdot 8 = 48$ square units.

6. Angles in isosceles triangle.

Since $XY = XZ$, the base angles are equal: $\angle Y = \angle Z = 55^\circ$.

$$\angle X = 180^\circ - 55^\circ - 55^\circ = 70^\circ.$$

7. Proof of converse.

Suppose $\angle B = \angle C$ in triangle ABC . Draw the angle bisector from A to side BC , meeting it at M .

In triangles ABM and ACM :

- $\angle ABM = \angle ACM$ (given)
- $\angle BAM = \angle CAM$ (angle bisector)
- $AM = AM$ (common side)

By AAS, $\triangle ABM \cong \triangle ACM$, so $AB = AC$. Thus the triangle is isosceles.

Homework

1. Find the area of a triangle with base 14 and height 9.
2. A triangular garden has area 120 m^2 and base 15 m. What is the height?
3. A rectangle has dimensions 8×6 . A diagonal divides it into two triangles. Find the area of each triangle.
4. Two triangles have the same area. One has base 10 and height 8. The other has base 16. What is its height?
5. For each set of information, determine if two triangles must be congruent:
 - (a) Three sides: 5, 12, 13 in both triangles
 - (b) Two sides 7 and 9, and the angle between them is 60° in both
 - (c) All three angles are 60° in both triangles
6. In isosceles triangle ABC with $AB = AC$, if $\angle B = 72^\circ$, find $\angle A$.
7. **M** In isosceles triangle DEF with $DE = DF = 13$ and base $EF = 10$, find:
 - (a) the height from D to EF
 - (b) the area
8. An isosceles triangle has a perimeter of 32 cm. If the base is 12 cm, find the length of each leg.
9. **M** In triangle ABC , M is the midpoint of BC . If $AB = AC$ and the area of $\triangle ABM$ is 15, what is the area of $\triangle ABC$?
10. **H** Prove that in any triangle, the sum of the areas of the two smaller triangles formed by a median equals the area of the original triangle, and that these two smaller triangles have equal areas.
11. **M** In triangle ABC , point D lies on BC such that $BD : DC = 2 : 3$. If the area of triangle ABC is 50, find the areas of triangles ABD and ADC .
Hint: Both triangles share the same height from A .
12. **H** In isosceles triangle ABC with $AB = AC$, points D and E lie on sides AB and AC respectively, such that $BD = CE$. Prove that $\triangle DBC \cong \triangle ECB$.
Hint: What do you know about $\angle DBC$ and $\angle ECB$?
13. **H** Triangles ABC and DBC share the same base BC . If the two triangles have equal areas, what must be true about points A and D ? Explain.
14. **H** In triangle ABC , D is the midpoint of AB and E is the midpoint of AC . If the area of triangle ABC is 40, find the area of triangle ADE .
Hint: Compare the base and height of $\triangle ADE$ to those of $\triangle ABC$.

Quick Check Answers

1. $\text{Area} = \frac{1}{2} \cdot 8 \cdot 5 = 20$.
2. $24 = \frac{1}{2} \cdot 6 \cdot h \Rightarrow h = 8$.
3. Yes, congruent by SAS (the angle $\angle B = \angle E$ is between the two given sides).
4. Yes, they can be different sizes. This is called **similarity**.
5. $\angle B = \angle C = \frac{180^\circ - 40^\circ}{2} = 70^\circ$.
6. $\angle R = \angle Q = 65^\circ$ (base angles), so $\angle P = 180^\circ - 65^\circ - 65^\circ = 50^\circ$.