

MATH 5: HANDOUT 15

GEOMETRY II: PARALLEL LINES AND TRIANGLES

Euclid's Axioms — The Rules of Geometry

By the time of Euclid (around 300 BC), geometry had already grown rich with discoveries. In his masterpiece *The Elements*, Euclid decided to organize all geometric knowledge in a completely logical way. He began with a few **definitions**, a few **common notions**, and only five simple **postulates** (axioms) — rules that describe how points and lines behave. From these, he proved every other statement about shapes, angles, and figures.

Euclid's Five Postulates (simplified wording).

Postulate 1. A straight line can be drawn joining any two points.

Postulate 2. A finite straight line can be extended continuously in a straight line.

Postulate 3. A circle can be drawn with any center and any radius.

Postulate 4. All right angles are equal to one another.

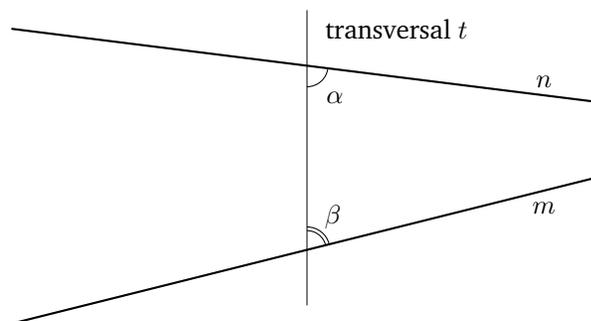
Postulate 5. If a straight line crossing two other straight lines makes the interior angles on the same side less than two right angles, the two lines — if extended indefinitely — meet on that side.

The first four postulates seem simple and natural. The fifth one, however, is surprisingly complicated. For more than two thousand years, mathematicians wondered whether it could be proven from the first four — but it cannot. The Fifth Postulate stands apart and gives birth to the very shape of the space we live in.

The Fifth Postulate and Parallel Lines

Let's restate Euclid's Fifth Postulate in modern language. Suppose a line t (called a **transversal**) crosses two other lines m and n . If the sum of the two **interior angles on the same side** of t is less than 180° , then the two lines m and n will meet on that side when extended.

That is, on the picture below, if $\alpha + \beta < 180^\circ$, then lines n and m will intersect on the side of angles α and β .



This postulate can also be stated in an equivalent and more familiar way:

Axiom

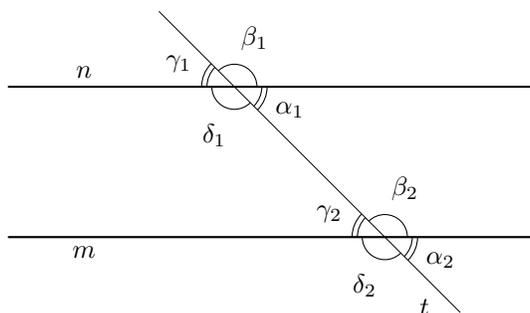
Parallel Postulate (modern form). Through a point not on a given line, there is exactly one line parallel to the given line.

Here, two lines are called **parallel** if they lie in the same plane and never meet, no matter how far they are extended.

The Parallel Postulate seems obvious, but it actually describes a deep property of space itself. If we change it, we get entirely different geometries — what we now call *non-Euclidean* geometries, which describe curved spaces and even the universe itself!

Angles Formed by a Transversal

When a transversal t crosses two lines m and n , several important pairs of angles are formed. Understanding how these angles relate is the key to reasoning about parallel lines.



Types of angles formed by a transversal:

- **Corresponding angles:** angles that occupy the same relative position at each intersection (α_1 and α_2 ; β_1 and β_2 , etc.)
- **Alternate interior angles:** angles that lie between the two lines, on opposite sides of the transversal (α_1 and γ_2 ; δ_1 and β_2)
- **Alternate exterior angles:** angles that lie outside the two lines, on opposite sides of the transversal (α_2 and γ_1 ; δ_2 and β_1)
- **Consecutive (same-side) interior angles:** angles on the same side of the transversal, between the two lines (α_1 and β_2 ; δ_1 and γ_2)

When the two lines are **parallel**, these angle pairs have special relationships.

Theorem

Theorem (Angle Relationships for Parallel Lines). If two parallel lines are cut by a transversal, then:

1. All corresponding angles are equal (e.g. $\alpha_1 = \alpha_2$).
2. All alternate interior angles are equal (e.g. $\alpha_1 = \gamma_2$).
3. All alternate exterior angles are equal (e.g. $\beta_1 = \delta_2$).
4. Consecutive interior angles are supplementary (add up to 180°) (e.g. $\alpha_1 + \beta_2 = 180^\circ$).

Sketch of proof. If corresponding (or alternate interior) angles were not equal, the transversal would make the sum of interior angles on one side either less or greater than two right angles.

By Euclid's Fifth Postulate, the lines would then meet — contradicting their parallelism.

□

Quick Check

1. Lines m and n are parallel. A transversal crosses them, forming an angle of 70° with line m . What is the measure of the corresponding angle at line n ?
2. If two parallel lines are cut by a transversal and one of the alternate interior angles is 115° , what is the other alternate interior angle?
3. Two parallel lines are cut by a transversal. If one of the consecutive interior angles is 65° , what is the other?

Consequences of the Parallel Postulate

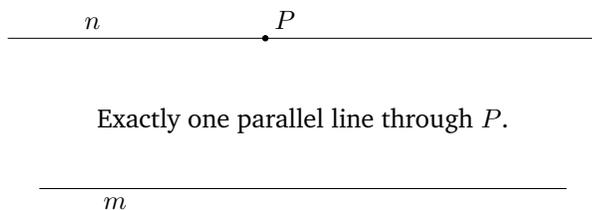
Parallel lines play a central role in geometry. From Euclid's Fifth Postulate, several important facts can be deduced. Each of these may seem obvious at first glance, but we will see that they can all be explained logically from the basic rules of geometry.

1. Through a given point not on a line, there is exactly one line parallel to it. This is in fact the modern restatement of Euclid's Fifth Postulate — sometimes called the **Parallel Postulate**.

Theorem

Statement. Given a line m and a point P not lying on it, there exists one and only one line through P that is parallel to m .

Reasoning. Through P , draw a transversal that intersects m . As the line through P rotates, the sum of the interior angles on the same side of the transversal changes. For one special position, that sum equals 180° ; in that position the lines do not meet. By Euclid's Fifth Postulate, any smaller sum would cause intersection on that side, and any larger sum would cause intersection on the opposite side. Hence, exactly one such position is parallel to m . \square

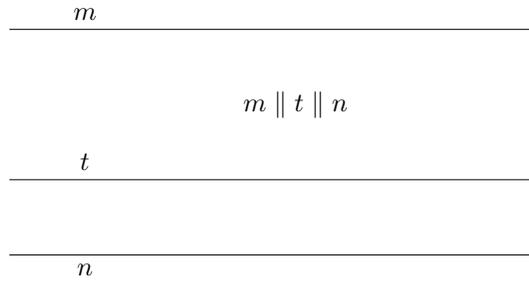


2. If two lines are both parallel to a third line, they are parallel to each other.

Theorem

Statement. If $m \parallel t$ and $n \parallel t$, then $m \parallel n$.

Proof. Let t be a line and m, n two other lines each parallel to t . Suppose, for contradiction, that m and n meet at some point A . Through A , there would then be two distinct lines (m and n) both parallel to t — violating the Parallel Postulate, which allows only one parallel through a given point. Hence, m and n cannot meet and must be parallel. \square

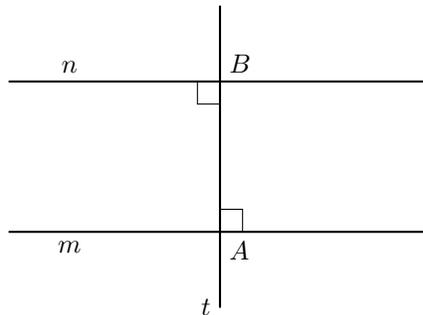


3. Two lines perpendicular to the same line are parallel.

Theorem

Statement. If two lines m and n are each perpendicular to a third line t , then m and n are parallel.

Proof. Let t be a line that intersects m at point A and n at point B . At each intersection, the line t forms right angles with both m and n . Thus, the alternate interior angles made by t with m and n are equal (each is 90°). By the converse of the Alternate Interior Angles Theorem, if a transversal makes equal alternate interior angles with two lines, the lines are parallel. Hence, $m \parallel n$. \square

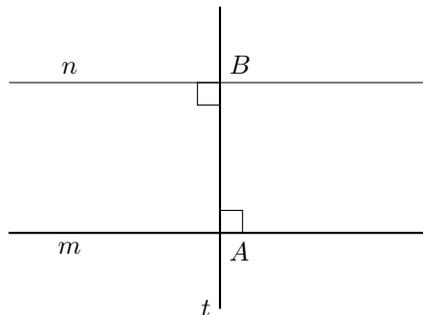


4. If a line is perpendicular to one of two parallel lines, it is also perpendicular to the other.

Theorem

Statement. If $m \parallel n$ and $t \perp m$, then $t \perp n$.

Proof. Let t be perpendicular to m . Thus, the angle between them is a right angle. When t crosses the second line n , it forms alternate interior angles with m . Since $m \parallel n$, these alternate interior angles are equal. Hence, the angle between t and n is also a right angle, and $t \perp n$. \square



Summary

- Through a given point, only one parallel line can be drawn to a given line.
- Lines parallel to the same line are parallel to each other.
- Lines perpendicular to the same line are parallel.
- A line perpendicular to one of two parallel lines is also perpendicular to the other.

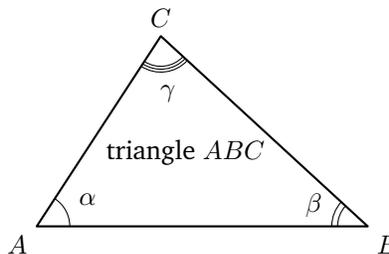
All these statements rely on Euclid's Fifth Postulate — the rule that determines how straight lines behave in a flat plane.

Quick Check

1. Line ℓ is parallel to line m , and line m is parallel to line n . What can you conclude about lines ℓ and n ?
2. Lines a and b are both perpendicular to line t . Are lines a and b parallel? Why or why not?
3. If $\ell \parallel m$ and line t is perpendicular to ℓ , what is the angle between t and m ?

Triangles and the Sum of the Angles

A **triangle** is the simplest closed figure formed by three straight segments joining three non-collinear points. The points are called the **vertices**, the segments are the **sides**, and the angles between the sides are the **interior angles** of the triangle.



Classifying Triangles (by sides).

- **Equilateral:** all sides equal.
- **Isosceles:** two sides equal.
- **Scalene:** all sides different.

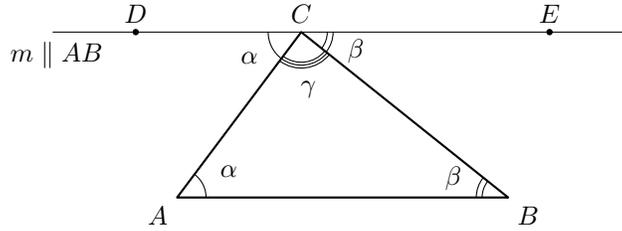
Classifying Triangles (by angles).

- **Acute:** all angles less than 90° .
- **Right:** one right angle.
- **Obtuse:** one angle greater than 90° .

Theorem

Theorem (Angle Sum Theorem). The sum of the interior angles of any triangle equals 180° .

Proof. Let ABC be any triangle. Through the vertex C , draw a line m parallel to the base AB .



Because line m is parallel to AB , two pairs of alternate interior angles are formed: $\angle CAB$ and $\angle DCA$; $\angle CBA$ and $\angle ECB$. Hence:

$$\angle DCA = \alpha, \quad \angle ECB = \beta.$$

At vertex C , the straight line m forms a straight angle:

$$\alpha + \gamma + \beta = 180^\circ.$$

Therefore, the sum of the three interior angles of triangle ABC equals 180° . □

$$\angle A + \angle B + \angle C = 180^\circ$$

Corollaries.

- A triangle cannot have more than one right or one obtuse angle, since their sum would already exceed 180° .
- The acute angles in a right triangle add up to 90° .
- If one angle of a triangle increases, the sum of the other two must decrease by the same amount.

Example 1. In triangle ABC , let $\angle A = 65^\circ$ and $\angle B = 45^\circ$. Then

$$\angle C = 180^\circ - (65^\circ + 45^\circ) = 70^\circ.$$

Summary

- A triangle is formed by three non-collinear points joined by segments.
- Through a vertex, we can draw a line parallel to the opposite side — a key step in many geometric proofs.
- The sum of the three interior angles of a triangle is always 180° .
- This result depends directly on the Parallel Postulate and holds only in Euclidean (flat) geometry.

Quick Check

1. In triangle ABC , $\angle A = 40^\circ$ and $\angle B = 75^\circ$. Find $\angle C$.
2. A triangle has angles x , $2x$, and $3x$. Find the value of x and all three angles.
3. Can a triangle have two right angles? Explain why or why not.
4. Classify a triangle with angles 60° , 60° , 60° by its sides and by its angles.

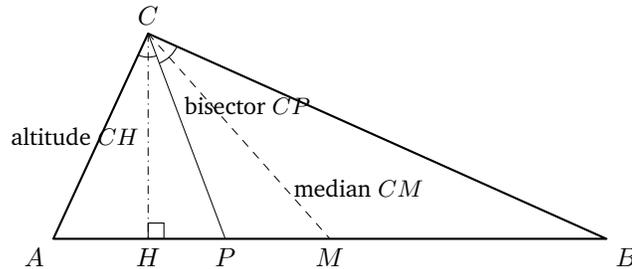
Special Lines in a Triangle

Inside any triangle we can draw several important lines from a vertex to the opposite side.

Median: A **median** is a segment from a vertex to the midpoint of the opposite side.

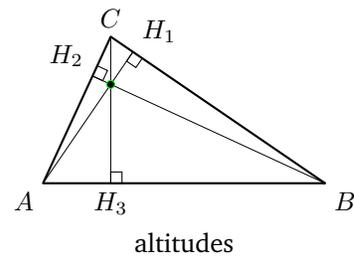
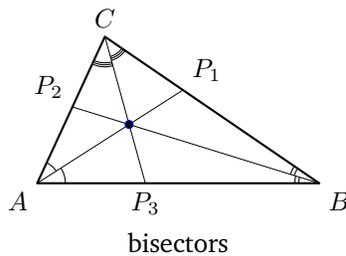
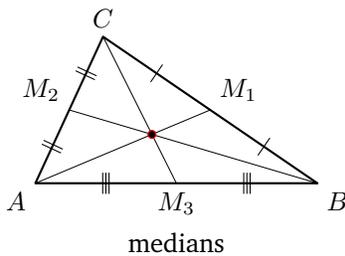
Bisector: An **angle bisector** is a ray from a vertex that divides the vertex angle into two equal angles.

Altitude: An **altitude** (or **height**) is a segment from a vertex perpendicular to the opposite side (or its extension).



Interesting Facts to Remember

- The three medians meet at one point (the *centroid*).
- The three angle bisectors meet at one point (the *incenter*).
- The three altitudes meet at one point (the *orthocenter*).



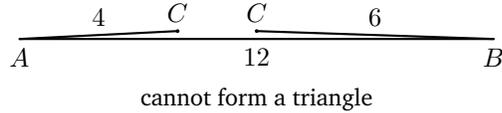
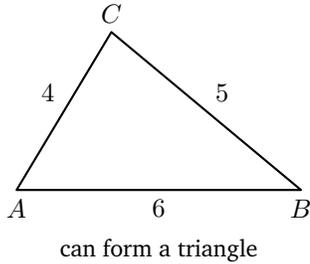
Quick Check

1. In triangle ABC , point M is the midpoint of BC . What is segment AM called?
2. In triangle PQR , segment PH is drawn from P perpendicular to QR . What is PH called?
3. What is the name of the point where all three medians of a triangle meet?
4. What is the name of the point where all three angle bisectors meet?

Triangle Inequality and Comparing Sides and Angles

Can we always make a triangle from three sides? Imagine we have three sticks of certain lengths. Can we always join their ends to form a triangle?

Let's experiment:



In the first case (sides 4, 5, 6), the ends meet perfectly — a triangle forms. In the second (4, 6, 12), the two shorter sides together are less than the longest one, so when we try to “close” the figure, the endpoints don’t meet — no triangle can form.

This shows that:

- If one side is *shorter* than the sum of the other two, a triangle *can* exist.
- If one side is *equal to or longer* than that sum, a triangle *cannot* exist.

Theorem

Triangle Inequality. In any triangle, the length of any side is less than the sum of the lengths of the other two sides.

$$AB < AC + CB, \quad AC < AB + BC, \quad BC < BA + AC.$$

Example 2. Examples.

- Can sides 3, 4, and 5 form a triangle? $3 + 4 > 5$, $3 + 5 > 4$, $4 + 5 > 3$ — all true \implies **Yes**.
- Can sides 2, 5, and 8 form a triangle? $2 + 5 = 7 < 8$ — fails \implies **No**.
- Can sides 6, 10, and 15 form a triangle? $6 + 10 = 16 > 15$ — OK; $6 + 15 = 21 > 10$; $10 + 15 = 25 > 6$ \implies **Yes**.

Comparing Sides and Angles in a Triangle. If we draw two different triangles, one with a very wide base and another with a narrow one, we quickly notice that the wider base faces the larger angle. This is not a coincidence — in every triangle, the size of an angle and the length of the side opposite to it are closely related. The greater the side, the greater the angle opposite it, and the smaller the side, the smaller the opposite angle.

Theorem

Bigger Side \iff Bigger Opposite Angle (statement). In a triangle, the longer side lies opposite the larger angle, and the larger angle lies opposite the longer side.

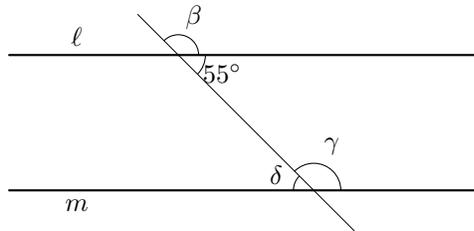
Again, we will use this powerful comparison principle; we will not prove it now.

Quick Check

1. Can the lengths 5, 7, and 11 form a triangle?
2. Can the lengths 3, 6, and 10 form a triangle?
3. Two sides of a triangle have lengths 8 and 12. What are the possible lengths for the third side?
4. In triangle XYZ , $XY = 5$, $YZ = 8$, and $XZ = 6$. Which angle is the largest?

Classwork

1. In the figure below, lines ℓ and m are parallel. Find the measures of angles β , γ , and δ .



2. Find the missing angle in each triangle:

- (a) $\angle A = 55^\circ$, $\angle B = 80^\circ$, $\angle C = ?$
- (b) $\angle P = 90^\circ$, $\angle Q = 27^\circ$, $\angle R = ?$
- (c) All three angles are equal.

3. Determine whether each set of side lengths can form a triangle. Explain your reasoning.

- (a) 4, 6, 9
- (b) 2, 4, 7
- (c) 5, 5, 10

4. Two sides of a triangle have lengths 5 and 9. If the third side has integer length, what are all possible values?

5. In triangle ABC , $\angle A = 75^\circ$, $\angle B = 45^\circ$, and $\angle C = 60^\circ$.

- (a) Which side is opposite to $\angle A$?
- (b) Which side is the longest?
- (c) Which side is the shortest?

6. In triangle DEF , point M is the midpoint of side EF , and point H is the foot of the perpendicular from D to EF . Identify segments DM and DH (median, altitude, or angle bisector?).

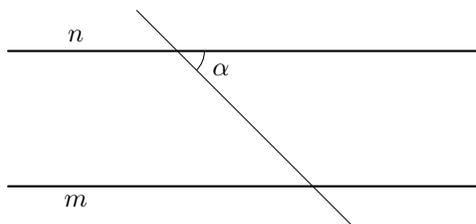
Classwork Solutions

1. Since $\ell \parallel m$:
 - β and the 55° angle are supplementary (they form a straight line), so $\beta = 180^\circ - 55^\circ = 125^\circ$.
 - γ and the 55° angle are alternate interior angles, so $\gamma = 55^\circ$.
 - δ and γ are supplementary, so $\delta = 180^\circ - 55^\circ = 125^\circ$.
2. (a) $\angle C = 180^\circ - 55^\circ - 80^\circ = 45^\circ$
(b) $\angle R = 180^\circ - 90^\circ - 27^\circ = 63^\circ$
(c) If all angles are equal: $3x = 180^\circ$, so $x = 60^\circ$. Each angle is 60° .
3. (a) Yes: $4 + 6 = 10 > 9$, $4 + 9 = 13 > 6$, $6 + 9 = 15 > 4$. All inequalities hold.
(b) No: $2 + 4 = 6 < 7$. The sum of two sides is less than the third side.
(c) No: $5 + 5 = 10$, which is not greater than 10. The inequality must be strict.
4. The third side must be greater than $9 - 5 = 4$ and less than $9 + 5 = 14$.
So the third side satisfies $4 < s < 14$.
The possible integer values are: 5, 6, 7, 8, 9, 10, 11, 12, 13 (9 values).
5. (a) Side BC is opposite to $\angle A$.
(b) The largest angle is $\angle A = 75^\circ$, so the longest side is BC (opposite to $\angle A$).
(c) The smallest angle is $\angle B = 45^\circ$, so the shortest side is AC (opposite to $\angle B$).
6.
 - DM is a **median** because M is the midpoint of side EF .
 - DH is an **altitude** because it is perpendicular to side EF .

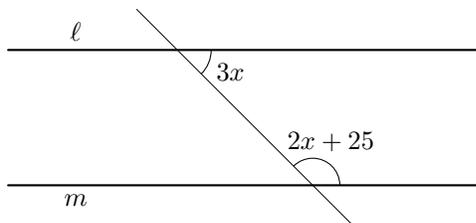
Homework

Difficulty: No mark = routine **M** = moderate **H** = challenging

1. In the figure below, lines m and n are parallel, and transversal t crosses them as shown. If $\alpha = 65^\circ$, find the measures of all the other angles.



2. Find the missing angle in each triangle:
- (a) $\angle A = 50^\circ$, $\angle B = 70^\circ$, $\angle C = ?$
 - (b) $\angle A = 90^\circ$, $\angle B = 35^\circ$, $\angle C = ?$
 - (c) $\angle A = \angle B = \angle C$ (an equilateral triangle)
3. Classify each triangle by its sides (equilateral, isosceles, or scalene) AND by its angles (acute, right, or obtuse):
- (a) A triangle with angles 60° , 60° , 60°
 - (b) A triangle with angles 90° , 45° , 45°
 - (c) A triangle with angles 100° , 50° , 30°
 - (d) A triangle with sides 5, 5, 6 and angles 67° , 67° , 46°
4. Determine whether each set of lengths can form a triangle. Explain your reasoning.
- (a) 3, 4, 5
 - (b) 1, 2, 3
 - (c) 7, 7, 12
 - (d) 5, 9, 15
 - (e) 8, 8, 8
 - (f) 2, 3, 6
5. Two sides of a triangle have lengths 7 and 10.
- (a) What is the smallest possible length for the third side? (It must be greater than what value?)
 - (b) What is the largest possible length for the third side? (It must be less than what value?)
 - (c) If the third side has integer length, list all possible values.
6. **M** In the figure, $\ell \parallel m$. Find the values of x and y .



7. The three angles of a triangle are x , $2x$, and $3x$. Find the measure of each angle. What type of triangle is this?

8. In triangle ABC , $\angle A = 80^\circ$, $\angle B = 60^\circ$, and $\angle C = 40^\circ$.
- Which side is the longest?
 - Which side is the shortest?
 - Order all three sides from shortest to longest.
9. In triangle PQR , point M is the midpoint of side QR , point H is the foot of the perpendicular from P to QR , and point S lies on QR such that $\angle QPS = \angle SPR$.
- What is segment PM called?
 - What is segment PH called?
 - What is segment PS called?
10. **M** In triangle ABC , side BC is extended to point D (so that C is between B and D). The angle $\angle ACD$ is called an *exterior angle* of the triangle.
- If $\angle A = 50^\circ$ and $\angle B = 60^\circ$, find $\angle ACD$.
 - Prove that the exterior angle $\angle ACD$ equals the sum of the two non-adjacent interior angles:
 $\angle ACD = \angle A + \angle B$.
Hint: Use the fact that $\angle ACB + \angle ACD = 180^\circ$.
11. **M** The lengths of two sides of a triangle are 8 and 15. The length of the third side is a positive integer. How many different triangles are possible?
12. **M** In triangle ABC , $\angle A = 70^\circ$. The angle bisector from A meets side BC at point D .
- What are the measures of $\angle BAD$ and $\angle CAD$?
 - If $\angle B = 60^\circ$, find $\angle ADB$ and $\angle ADC$.
Hint: Use the angle sum in triangles ABD and ACD .
13. **H** **Challenge.** In triangle ABC , the three altitudes meet at point H (the orthocenter). If $\angle A = 70^\circ$ and $\angle B = 50^\circ$, find $\angle AHB$.
Hint: Look at the quadrilateral formed by A , the foot of the altitude from A , H , and the foot of the altitude from B .

Quick Check Answers

Quick Check 1 (Angles Formed by a Transversal):

1. 70° (corresponding angles are equal)
2. 115° (alternate interior angles are equal)
3. $180^\circ - 65^\circ = 115^\circ$ (consecutive interior angles are supplementary)

Quick Check 2 (Consequences of the Parallel Postulate):

1. $\ell \parallel n$ (lines parallel to the same line are parallel to each other)
2. Yes, lines perpendicular to the same line are parallel
3. 90° (a line perpendicular to one of two parallel lines is perpendicular to the other)

Quick Check 3 (Triangles and the Sum of the Angles):

1. $\angle C = 180^\circ - 40^\circ - 75^\circ = 65^\circ$
2. $x + 2x + 3x = 180^\circ \Rightarrow 6x = 180^\circ \Rightarrow x = 30^\circ$. The angles are 30° , 60° , 90° .
3. No. Two right angles would sum to 180° , leaving 0° for the third angle.
4. Equilateral (all angles equal implies all sides equal) and acute (all angles less than 90°)

Quick Check 4 (Special Lines in a Triangle):

1. Median
2. Altitude (or height)
3. Centroid
4. Incenter

Quick Check 5 (Triangle Inequality):

1. Yes: $5 + 7 = 12 > 11$, $5 + 11 = 16 > 7$, $7 + 11 = 18 > 5$
2. No: $3 + 6 = 9 < 10$
3. The third side must be greater than $12 - 8 = 4$ and less than $12 + 8 = 20$, so $4 < s < 20$.
4. $\angle X$ is the largest because it is opposite the longest side $YZ = 8$.