# MATH 5: HANDOUT 14

## GEOMETRY I: BUILDING BLOCKS

## 1 Building Blocks of Geometry

A **definition** is a statement that explains the meaning of a term, word, or concept. In mathematics, precise language matters: before we can talk about an idea, we must agree what it means. For example, before we can study *triangles*, we must first know what a *side*, a *vertex*, and an *angle* are.

Sometimes, however, there are ideas so basic that they cannot be defined in simpler terms. In geometry, we will simply *accept* some ideas as **undefined terms** — we use them without trying to define them through others. The words *point*, *line*, and *plane* are examples of such fundamental notions. All other geometric objects (segments, rays, triangles, circles, etc.) are built from them.

**Example**

**Example.** To define a *segment*, we must already understand what a *line* and a *point* are. Hence, "point" and "line" must come first — they cannot depend on the segment for their definition.

**Axioms and Theorems.** Once the undefined terms are accepted, we can state **axioms** (also called **postulates**) — basic rules about how these objects behave. An *axiom* is something we agree to be true without proof; it serves as a starting point for logical reasoning.

**Axiom**

**Example of an Axiom:** *Through any two distinct points, one and only one straight line can be drawn.*

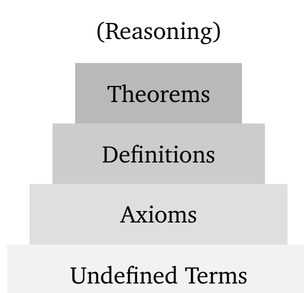
From axioms and definitions, mathematicians then prove new facts called **theorems**. A theorem is a statement that has been logically shown to be true, based on what we already accept as true.

**Theorem**

**Example of a Theorem:** Vertical angles are equal. (This result can be proven using the idea that adjacent angles on a straight line are supplementary.)

**How Mathematics Is Built.** Mathematics is like a tall building:

- **Undefined terms** are the foundation stones.
- **Axioms** are the main rules that describe how the foundation behaves.
- **Definitions** explain new ideas in terms of old ones.
- **Theorems** are the walls and towers we build by reasoning carefully from what we already know.



**Why Axioms Matter.** Axioms cannot be "proven" because they are our starting assumptions. But we choose them carefully so that they seem natural and consistent with our everyday experience of space. Once axioms are accepted, we can logically prove theorems and be certain they are true within that system.

**Key Takeaways**

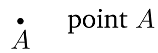
**Definitions** explain meaning. **Axioms** state what we accept without proof. **Theorems** are what we prove using logic and the axioms.

## 2 The Undefined Terms of Geometry

Before we can build any structure, we need a foundation. For geometry, this foundation begins with a few words that we cannot truly define, but only describe through intuition and examples. They are the **undefined terms** — the primitive ideas upon which everything else rests.

The three basic undefined terms are the **point**, the **line**, and the **plane**. Every other object in geometry — segments, angles, triangles, circles — is defined using these three.

**Point.** Imagine marking a tiny dot on your paper with a sharp pencil. That dot represents a *point*, but the point itself has no size — no length, width, or thickness. It is simply a precise location in space, not a little circle or speck of graphite. In our drawings, we use a small dot to *represent* it, but in our mind we must remember that the true point has no dimensions at all. We usually label points with **capital letters** such as $A, B$, or $C$.

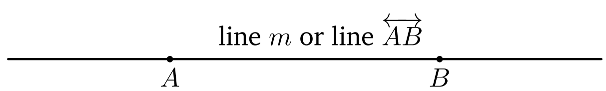



In everyday life, you can think of a point as a single location on a map, a tiny star in the sky, or the tip of a very fine needle.

**Line.** Now imagine stretching a perfectly straight thread so that it extends endlessly in both directions. No matter how far you go, it never stops and never bends. That is a **line**: it has length but no thickness or width. A line is one-dimensional — it lives in one direction only.

We represent a line on paper as a straight path with arrowheads at both ends, to show that it goes on forever. A line can be named:

- by a single lowercase letter, for example line $m$; or
- by any two points on it, such as $\overleftrightarrow{AB}$.

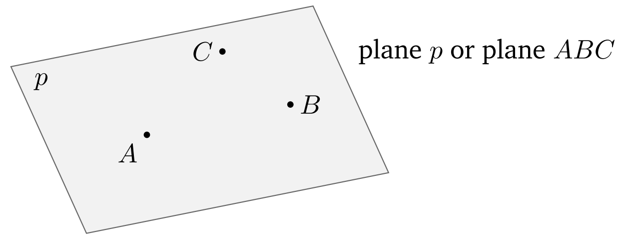



Through two points we can always draw one and only one straight line — a rule that will soon become our first axiom.

**Plane.** Next, imagine placing a sheet of paper on your desk. Now imagine that the paper extends endlessly in every direction — no edges, no corners, just an infinite flat surface. That is what mathematicians call a **plane**. A plane has length and width, but no thickness. Although we often draw it as a tilted parallelogram with edges, we must remember that in geometry the plane continues forever and has no boundaries.

A plane can be named:

- by a single lowercase letter, for example plane $p$; or
- by three non-collinear points (points not on the same line), such as plane $ABC$.




Every line and every point we draw will be thought of as lying *in some plane*. Later we will see that three points always determine exactly one plane, provided they do not all lie on a single line.

**Axiom**

**Axiom of the Straight Line.** Through any two distinct points, one and only one straight line can be drawn.

This statement seems obvious, yet it is not something we can prove. We accept it as one of the basic truths about how points and lines behave — an *axiom* that guides the entire structure of geometry. From it, and other axioms like it, all the great theorems of geometry will follow.

## 3 Segments, Rays, and Angles

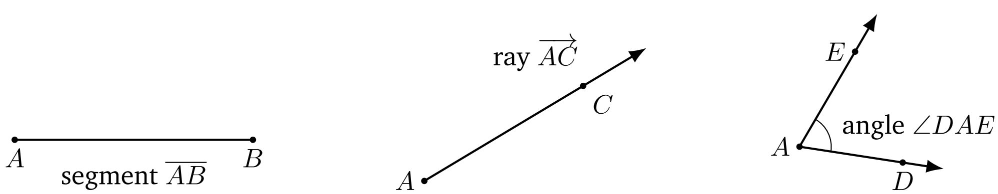
Up to now, we have talked about points, lines, and planes — the basic building blocks of geometry. By combining these, we can form new figures. When we take only a *part* of a line or look at two lines meeting at a point, we get the most common geometric objects: **segments**, **rays**, and **angles**. These shapes appear everywhere — in triangles, polygons, and circles — so it is worth understanding them carefully.

**Segments.** If we mark two distinct points $A$ and $B$ on a straight line, the set of all points between them is called a **line segment**. The points $A$ and $B$ are its *endpoints*, and we write the segment as $\overline{AB}$. Unlike a line, a segment has a definite length — it begins at $A$ and ends at $B$.

**Rays.** Suppose we fix one endpoint $A$ on a line, and then choose another point $B$ on that line. If we start at $A$ and include every point that lies on the same side of $A$ as $B$, we get a **ray**. The ray starts at $A$ and extends infinitely through $B$. We name it by its endpoint and another point on it: $\overrightarrow{AB}$.

**Angles.** When two rays share the same endpoint, they form an **angle**. The common endpoint is called the **vertex** of the angle, and the rays are its **sides**. Angles are named by three capital letters with the vertex in the middle (for example, $\angle DAE$), or sometimes by a Greek letter such as $\alpha$ or $\beta$.

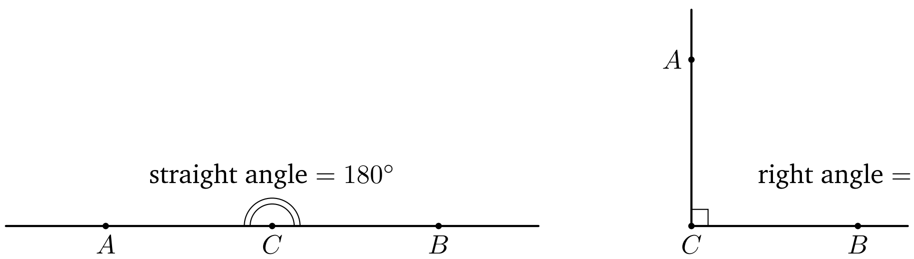
Angles describe how "open" two rays are from each other — in other words, how much one ray must *turn* to reach the other. To express this amount of turning, we need a way to measure angles.



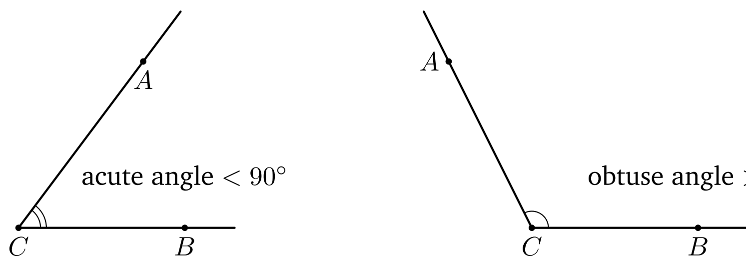
**Measuring Angles.** Long ago, people noticed that a full turn around a point — one complete rotation — could be conveniently divided into $360$ equal parts. Each part is called a **degree**, written with the symbol $^\circ$.

- A complete rotation measures $360^\circ$.
- Half of that rotation, when the two rays form a straight line, measures $180^\circ$ — a **straight angle**.
- One quarter of a full turn measures $90^\circ$ — a **right angle**.

Smaller angles are measured in degrees as well. For example, an angle of $45^\circ$ is half of a right angle.




Two angles are called **supplementary** if together they make a straight angle (their measures add up to $180^\circ$). If an angle equals its own supplement, it must measure $90^\circ$; we call such an angle a **right angle**. Angles smaller than $90^\circ$ are called **acute**, and angles larger than $90^\circ$ but smaller than $180^\circ$ are called **obtuse**.




**Example**

**Example (Straight angle).** If point $C$ lies on line $AB$, then the rays $\overrightarrow{CA}$ and $\overrightarrow{CB}$ form a straight angle at $C$, measuring $180^\circ$.

**Greek Letters in Geometry**

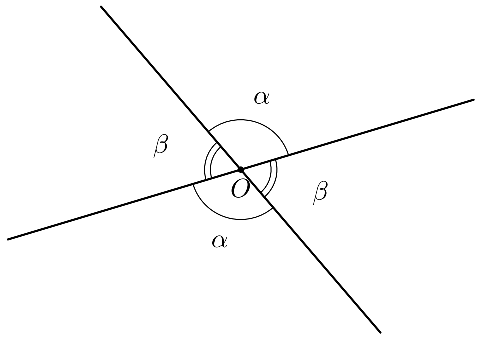
In geometry, we often use small Greek letters to name unknown angles or sides: $\alpha$ (alpha), $\beta$ (beta), $\gamma$ (gamma), $\delta$ (delta), and so on.

| | | | |
|---|---|---|---|
| $\alpha$ | alpha | $\beta$ | beta |
| $\gamma$ | gamma | $\delta$ | delta |
| $\theta,\ \vartheta$ | theta | $\varphi,\ \phi$ | phi |
| $\psi$ | psi | $\omega$ | omega |

We use them because they are simple, elegant, and help avoid confusion with points (which use Latin capitals) and lines (which use Latin lowercase letters).

# 4 Vertical Angles

When two lines intersect, they form four angles. Opposite (non-adjacent) angles are called **vertical angles**.




**Theorem**

**Vertical Angles Theorem.** Vertical angles are equal.

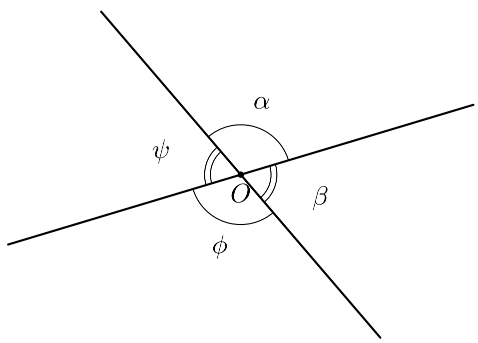
*Proof.* Let the intersecting lines form angles $\alpha, \beta, \varphi, \psi$ around the vertex. Each pair of adjacent angles is supplementary, so

$$\alpha + \psi = 180^\circ \quad \text{and} \quad \alpha + \beta = 180^\circ.$$

Hence

$$\psi = 180^\circ - \alpha \quad \text{and} \quad \beta = 180^\circ - \alpha.$$

As we can see, $\psi = \beta$. Similarly, the other pair is equal ($\varphi = \alpha$). Hence, vertical angles are equal. □




**Example**

**Question.** If two angles are equal, must they be vertical angles?
**Answer.** Not necessarily—two right angles in different places are equal but not vertical. Verticality depends on being opposite at the *same* intersection.

# 5 Guided Practice

1. Draw one point $A$. How many distinct straight lines pass through $A$?

2. Draw two points $B, C$. How many distinct straight lines pass through both $B$ and $C$?

3. On a line, mark points $D$ and $E$. Identify the segment $\overline{DE}$, and the rays $\overrightarrow{DE}$ and $\overrightarrow{ED}$.

**4.** Construct $\angle FGH$ and label its vertex. Which rays form this angle?

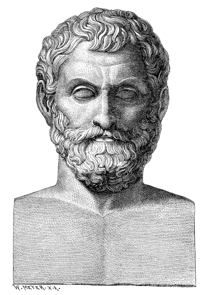
**5.** Create two intersecting lines. Label all four angles and circle the two vertical angles. Explain why they are equal.

**6.** If $\angle XOY$ is a straight angle and $\angle XOZ = 38°$, find the measure of $\angle ZOY$. Are $\angle XOZ$ and $\angle ZOY$ supplementary?

**7.** True or false: If two angles are supplementary and equal, then each is a right angle. Explain.

## 6 Historical Note — The Birth of Geometry in Ancient Greece

Long before rulers and protractors, people already used geometry — the Egyptians measured fields after the Nile floods, and the Babylonians built right angles in their temples. But it was in **ancient Greece** that geometry changed from a craft into a science of reasoning.



Thales of Miletus

**Thales of Miletus (624–546 BC).** Thales is often called the *first mathematician*. He visited Egypt and observed that surveyors measured equal angles each time two lines crossed. Thales realized that this was not a coincidence — he reasoned that *vertical angles are always equal*, and proved it logically, without measurement. This was a revolutionary idea: that one can discover truth by reasoning, not by experiment alone.

Euclid of Alexandria

**Euclid of Alexandria (around 300 BC).** A few centuries later, Euclid gathered all known geometric knowledge into a monumental work called *The Elements*. In it, he began with only a few simple **undefined terms** (point, line, plane) and a short list of **axioms**, then used logic to prove hundreds of **theorems**. Euclid's method of building an entire science from basic assumptions became the model for all of mathematics.

**Legacy.** For over two thousand years, geometry was taught from Euclid's *Elements*. The language and reasoning we still use — axioms, proofs, and even the diagrams with letters $A, B, C$ — come directly from this Greek tradition.

**Key Takeaways**

The Greeks turned geometry into a language of reasoning. They showed that from a few clear ideas — points, lines, planes, and a handful of axioms — an entire world of truths can be discovered by pure thought.

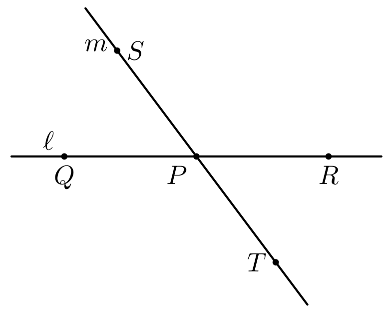
**Summary**

- **Undefined terms:** point, line, plane.
- **Definitions:** segment, ray, angle, straight angle, supplementary angles, right angle, vertical angles.
- **Key measures:** straight angle $180°$; right angle $90°$.
- **Axiom:** Through any two points, exactly one line.
- **Theorem:** Vertical angles are equal.

## Homework

1. **(Basic Construction)** Draw three points $A$, $B$, and $C$ that do not all lie on the same line (such points are called *non-collinear*). Then using a straightedge, draw all possible lines through pairs of these points.
   (a) How many distinct lines did you draw?
   (b) Could you draw more lines using only these three points? Explain why or why not, using the axiom from the handout.

2. **(Naming Geometric Objects)** Consider the figure below, where lines $\ell$ and $m$ intersect at point $P$, and points $Q$, $R$, $S$, $T$ lie on the lines as shown.



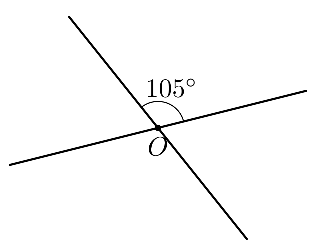

   Identify each of the following:
   (a) Two different segments
   (b) Two different rays that have endpoint $P$
   (c) The vertex of $\angle QPS$
   (d) A pair of vertical angles (name them using three letters each)

3. **(Classifying Angles)** Classify each angle as **acute**, **right**, **obtuse**, or **straight**:

| | | |
|---|---|---|
| (a) $47°$ | (c) $128°$ | (e) $89°$ |
| (b) $90°$ | (d) $180°$ | (f) $91°$ |

4. **(Supplementary Angles)**
   (a) Two angles are supplementary. One angle measures $67°$. What is the measure of the other angle?
   (b) Two angles are supplementary. One angle is three times as large as the other. Find the measure of each angle.

5. **(Vertical Angles)** Two lines intersect at point $O$. One of the four angles formed measures $105°$.




   Find the measures of the other three angles. Explain your reasoning using the vertical angles theorem and the fact that adjacent angles on a line are supplementary.

6. **(Algebraic Reasoning)** Two supplementary angles have measures that differ by $24°$. Let the smaller angle have measure $x$.

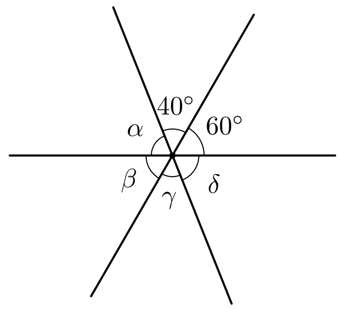
(a) Write an equation involving $x$ that expresses the fact that the two angles are supplementary.

(b) Solve the equation to find $x$.

(c) What are the measures of both angles?

(d) Classify each angle as acute, right, or obtuse.

**7. (Three Lines Through a Point)** Three lines pass through the same point $O$, forming six angles around $O$ as shown. Two adjacent angles measure $40°$ and $60°$ as indicated.




(a) Find the measures of $\alpha$, $\beta$, $\gamma$, and $\delta$.

(b) Verify that all six angles sum to $360°$.

**8. (True or False)** Determine whether each statement is **true** or **false**. If false, provide a counterexample or explain why.

(a) Two distinct points determine exactly one line.

(b) A ray has two endpoints.

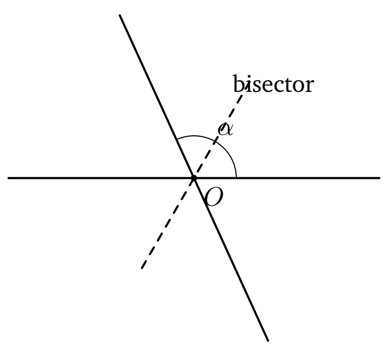
(c) If two angles are equal, they must be vertical angles.

(d) All right angles are equal to each other.

(e) If two lines intersect, they intersect at exactly one point.

(f) Three points always determine exactly one plane.

**9. (Angle Bisector)** Two lines intersect at point $O$, forming four angles. Let one of the angles have measure $\alpha$. A third line through $O$ *bisects* this angle (divides it into two equal parts).




(a) In terms of $\alpha$, what are the measures of the two angles created when the bisector divides angle $\alpha$?

(b) The original two lines created four angles. The bisector now creates six angles around $O$. Express all six angle measures in terms of $\alpha$.

(c) Verify that your six angles sum to $360°$.

(d) If $\alpha = 70°$, find the numerical value of each of the six angles.

**10. (Challenge: Two Bisectors)** Two lines intersect at point $O$, forming four angles. Two of these angles are vertical (opposite) angles, each measuring $\alpha$. The other two vertical angles each measure $\beta$.

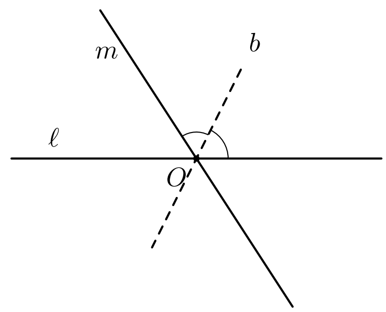
Now suppose we draw the angle bisector of one angle measuring $\alpha$, *and* the angle bisector of one angle measuring $\beta$ (these bisectors are different lines).

(a) What is the relationship between $\alpha$ and $\beta$? Express $\beta$ in terms of $\alpha$.

(b) When both bisectors are drawn, eight angles are formed around $O$. Find the measure of each angle in terms of $\alpha$.

(c) Prove that the two bisectors are perpendicular to each other (i.e., they meet at a $90°$ angle). *Hint: Find the angle between the two bisectors using your answer to part (b).*

**11. (Challenge: A Theorem About Angle Bisectors)** Let two lines $\ell$ and $m$ intersect at point $O$. Let the angle bisector of one of the four angles be line $b$.




**Prove:** The line $b$ also bisects the vertical angle (the angle opposite to the one it was constructed to bisect).

*Hint: Let the original angle measure $2\theta$ (so each half measures $\theta$). Use the vertical angles theorem and the fact that supplementary angles sum to* $180°$.

**12. (Exploration: Four Lines Through a Point)** Four distinct lines pass through a single point $O$, creating eight angles around $O$.

(a) What is the sum of all eight angles? Justify your answer.

(b) If all eight angles are equal, what is the measure of each angle?

(c) Is it possible for exactly two of the eight angles to be right angles? Explain why or why not.

(d) Is it possible for exactly three of the eight angles to be right angles? Explain why or why not.

(e) Is it possible for exactly three of the eight angles to be equal? Explain why or why not.

(f) Is it possible for exactly four of the eight angles to be equal? Explain why or why not.

**13. (Challenge: Definitions and Logical Reasoning)** Consider the following proposed definitions and decide whether each is a **good definition** (precise and unambiguous) or a **bad definition** (vague, circular, or problematic). Explain your reasoning.

(a) "A *point* is a very small dot."

(b) "A *line segment* is the set of all points between two endpoints $A$ and $B$, together with $A$ and $B$ themselves."

(c) "An *acute angle* is an angle that looks sharp."

(d) "*Vertical angles* are the two non-adjacent angles formed when two lines intersect."

(e) "A *right angle* is an angle whose measure equals that of its supplement."