MATH 5: HANDOUT 13

IRRATIONAL NUMBERS AND ROOTS

Irrational Numbers and Square Roots

Rational and Irrational Numbers

A **rational number** is a number that can be written as a ratio of two integers:

$$a = \frac{p}{q}, \quad p \in \mathbb{Z},\ q \in \mathbb{N}.$$

Examples include:

$$\frac{1}{2} = 0.5, \quad \frac{1}{3} = 0.\overline{3}, \quad -\frac{7}{4} = -1.75.$$

Every rational number has a decimal representation that is either **finite** (ends) or **periodic** (repeats).

$$\frac{1}{8} = 0.125 \quad \text{(finite)}, \qquad \frac{2}{11} = 0.\overline{18} \quad \text{(periodic)}.$$

Numbers that cannot be expressed as $\frac{p}{q}$ are called **irrational**. Their decimal expansion is infinite and non-repeating:

$$0.101001000100001\ldots \quad \text{or} \quad 0.12345678910111213\ldots$$

Famous examples include:

$$\sqrt{2} = 1.4142135\ldots, \quad \pi = 3.1415926\ldots, \quad e = 2.7182818\ldots$$

Irrational numbers fill the gaps between rational ones — they are "uncountably many"! Between any two fractions there are infinitely many irrational numbers.

Quick Check

Decide whether each number is **rational** or **irrational**. Briefly explain your reasoning.

1. 0.375
2. $0.\overline{27}$
3. $\sqrt{2}$
4. π
5. $\frac{22}{7}$

What is a Square Root?

The **square root** of a number a is a number whose square is a:

$$x^2 = a.$$

Every positive real number a has two square roots — one positive and one negative. The positive one is called the **arithmetic (principal) square root** and is written as $\sqrt{a}$.

$$\sqrt{25} = 5, \qquad \text{but the equation } x^2 = 25 \text{ has two solutions: } x = \pm 5.$$

Basic examples:

$$\sqrt{0} = 0, \quad \sqrt{1} = 1, \quad \sqrt{9} = 3, \quad \sqrt{\tfrac{1}{64}} = \tfrac{1}{8}, \quad \sqrt{\tfrac{36}{25}} = \tfrac{6}{5}.$$

Quick Check

Answer each question.

1. What does $\sqrt{a}$ mean in words?
2. How many solutions does the equation $x^2 = 16$ have?
3. Is $\sqrt{-4}$ a real number? Why or why not?
4. Circle all numbers that are **perfect squares**.

$$2, \quad 4, \quad 8, \quad 9, \quad 12, \quad 16, \quad 20, \quad 25$$

Historical Note: Hippasus and the Discovery of Irrational Numbers

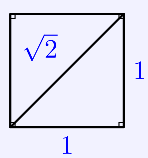
Around the 5^{th} century BCE, the Pythagorean school in ancient Greece believed that "everything in the universe can be expressed through whole numbers and their ratios."

One of their members, **Hippasus of Metapontum**, studied the diagonal of a square with side 1. He found that its length d satisfies $d^2 = 1^2 + 1^2 = 2$, so $d = \sqrt{2}$.

When he tried to express $\sqrt{2}$ as a ratio of two integers, he discovered that it was impossible — the side and diagonal of a square have no common measure. In modern terms, $\sqrt{2}$ is **irrational**.

This was a shocking revelation: it showed that not all quantities can be written as fractions. According to legend, the Pythagoreans were so disturbed by this result that they cast Hippasus overboard for revealing a truth that contradicted their belief that "all is number."

Hippasus of Metapontum



Key properties

- $(\sqrt{a})^2 = a$
- $\sqrt{ab} = \sqrt{a}\sqrt{b}$ for $a, b \geq 0$
- $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$
- For $a > 0$, equation $x^2 = a$ has two solutions: $\pm\sqrt{a}$

Simplifying Square Roots

In many problems, we encounter square roots of numbers such as $\sqrt{20}$ or $\sqrt{72}$. These expressions are correct, but they are often not written in their simplest form.

The goal of *simplifying a square root* is to rewrite it in a form where:

- all perfect squares have been taken out of the square root, and
- the number left inside the square root is as small as possible.

To do this, we use the identity

$$\sqrt{ab} = \sqrt{a}\sqrt{b}, \qquad \text{for } a, b \geq 0.$$

This identity allows us to factor the number inside the square root and separate any perfect square factors, which can then be simplified.

Simplifying square roots makes expressions easier to compare, combine, and use in further calculations, especially in algebra, geometry, and later topics such as quadratic equations.

Example 1. $\sqrt{20}$
Factor the number under the square root:

$$20 = 4 \cdot 5.$$

Apply the identity:

$$\sqrt{20} = \sqrt{4 \cdot 5} = \sqrt{4}\sqrt{5}.$$

Since $\sqrt{4} = 2$, we obtain

$$\sqrt{20} = 2\sqrt{5}.$$

Example 2. $\sqrt{50}$
Factor 50:

$$50 = 25 \cdot 2.$$

Then

$$\sqrt{50} = \sqrt{25 \cdot 2} = \sqrt{25}\sqrt{2}.$$

Since $\sqrt{25} = 5$, we get

$$\sqrt{50} = 5\sqrt{2}.$$

Example 3. $\sqrt{72}$
Factor 72:

$$72 = 36 \cdot 2.$$

Thus

$$\sqrt{72} = \sqrt{36 \cdot 2} = \sqrt{36}\sqrt{2}.$$

Since $\sqrt{36} = 6$, we conclude

$$\sqrt{72} = 6\sqrt{2}.$$

Example 4. $\sqrt{45}$
Factor 45:

$$45 = 9 \cdot 5.$$

Then

$$\sqrt{45} = \sqrt{9 \cdot 5} = \sqrt{9}\sqrt{5}.$$

Since $\sqrt{9} = 3$, we get

$$\sqrt{45} = 3\sqrt{5}.$$

Quick Check

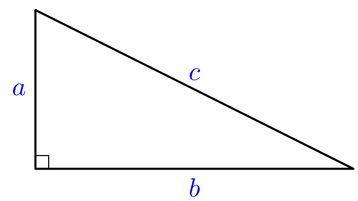
Simplify each square root.

1. $\sqrt{12}$
2. $\sqrt{27}$
3. $\sqrt{75}$

Connection to Geometry: Pythagorean Theorem

In any right triangle with legs a and b and hypotenuse c, the following relation (the **Pythagorean Theorem**) holds:

$$a^2 + b^2 = c^2.$$



This theorem allows us to compute the length of one side of a right triangle if the other two are known.

Example 1: If a right triangle has legs 3 and 4, then

$$c^2 = 3^2 + 4^2 = 9 + 16 = 25 \quad \Rightarrow \quad c = 5.$$

It is possible — but not easy — to find more right triangles where all sides are whole numbers. The triangle with sides $3, 4, 5$ is the most famous such triangle.

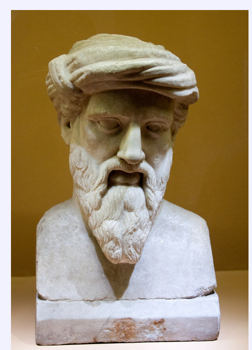
Example 2: If a right triangle has a leg 10 and the hypothenuse 11, then

$$11^2 = 10^2 + b^2 \quad \Rightarrow \quad b^2 = 121 - 100 = 21 \quad \Rightarrow \quad b = \sqrt{21}.$$

Who Was Pythagoras?

Pythagoras of Samos (around 570–495 BCE) was a **Greek philosopher, scientist, and mathematician**, best known for the **Pythagorean Theorem**.

He founded a secretive philosophical school in Croton, southern Italy, where his followers — the **Pythagoreans** — lived almost like monks. They studied mathematics, music, and astronomy, followed strict rules, and believed that the universe could be understood entirely through **numbers and harmony**.



Pythagoras of Samos

Pythagoras taught that numbers had **mystical properties**: the number 1 represented unity, 2 diversity, 3 harmony, and 10 perfection. He and his students discovered that musical notes pleasing to the ear come from strings whose lengths form simple **ratios of whole numbers** — a striking link between math and music.

Their community had unusual customs. They were said to believe in **reincarnation**, to live mostly on a vegetarian diet, and even to **avoid eating beans** — possibly for symbolic or ritual reasons. Members took vows of secrecy, and breaking them could lead to expulsion.

Although legend and fact are hard to separate, Pythagoras' ideas deeply shaped later Greek thought. He inspired the belief that **mathematics reveals the hidden order of the cosmos** — a view that influenced Plato, Kepler, and many later scientists and philosophers.

Quick Check

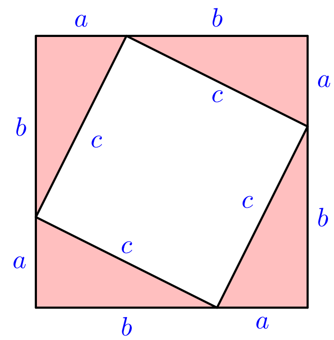
In the examples below, a and b are legs of the right angle triangle, and c is a hypothenuse. Find the missing side:

1. $a = 5,\ b = 12,\ c = ?$
2. $a = 6,\ b = ?,\ c = 10$

Proof of the Pythagorean Theorem

There are many proofs of the Pythagorean Theorem. Here is one of the most famous, using areas. Five more proofs are given below.

Step 1. Construct a square of side length $a + b$. Inside it, arrange four congruent right triangles with legs a and b and hypotenuse c as shown below.



Step 2. The large square has area

$$(a+b)^2.$$

It is composed of four right triangles (each of area $\frac{1}{2}ab$) and a smaller inner square of side c, area c^2. So the total area is also

$$4 \cdot \frac{1}{2}ab + c^2 = 2ab + c^2.$$

Step 3. Equating the two expressions for the area gives:

$$(a+b)^2 = 2ab + c^2.$$

Note, that

$$(a+b)^2 = (a+b)(a+b) = a^2 + ab + ba + b^2 = a^2 + 2ab + b^2.$$

Now, we can rewrite the equality above and simplify it:

$$a^2 + 2ab + b^2 = 2ab + c^2 \quad \Rightarrow \quad a^2 + b^2 = c^2.$$

This completes the proof. ■

For a square with side 1, the diagonal is $\sqrt{2}$. But what is $\sqrt{2}$ exactly? Let's investigate.

Why $\sqrt{2}$ is Irrational

Assume, for contradiction, that $\sqrt{2}$ is rational:

$$\sqrt{2} = \frac{p}{q}, \quad \text{where } \frac{p}{q} \text{ is reduced.}$$

Then

$$\left(\frac{p}{q}\right)^2 = 2 \Rightarrow p^2 = 2q^2.$$

So p^2 is even $\Rightarrow$ p is even: $p = 2p_1$. Substitute:

$$4p_1^2 = 2q^2 \Rightarrow q^2 = 2p_1^2,$$

which means q is even too. Then both p and q are even — contradiction! Therefore, $\sqrt{2}$ cannot be written as a fraction.

$$\boxed{\sqrt{2} \text{ is irrational.}}$$

The discovery that $\sqrt{2}$ cannot be written as a fraction was only the beginning. Once mathematicians understood this, they started asking:

"What about $\sqrt{3}$, $\sqrt{5}$, or $\sqrt{6}$?"

It turns out that the same kind of reasoning works for many other numbers.

Most square roots are irrational!

If a number a is not a **perfect square**, then $\sqrt{a}$ is **irrational**.
A **perfect square** is any number that equals n^2 for some integer n:

$$1, 4, 9, 16, 25, 36, 49, 64, 81, 100, \ldots$$

So:

$$\sqrt{4} = 2 \text{ (rational)},$$
$$\sqrt{9} = 3 \text{ (rational)},$$
$$\sqrt{3},\ \sqrt{5},\ \sqrt{6},\ \sqrt{7},\ \sqrt{8} \text{ are all irrational.}$$

In fact, there are infinitely many irrational square roots — and even more astonishingly, there are infinitely many irrational numbers *between* any two rational ones! The real number line is a seamless blend of both kinds, with rational points sprinkled among an uncountable sea of irrationals.

Approximating $\sqrt{2}$

Since $1^2 = 1$ and $2^2 = 4$, the number $\sqrt{2}$ must lie between 1 and 2.

$$1 < \sqrt{2} < 2.$$

Let us try to narrow this interval step by step by squaring decimal guesses.

$$\begin{array}{lcll} 1.3^2 & = & 1.69 & \text{too small (need a larger number)} \\ 1.4^2 & = & 1.96 & \text{still a bit less than 2} \\ 1.5^2 & = & 2.25 & \text{too large} \end{array}$$
$$\Rightarrow \quad 1.4 < \sqrt{2} < 1.5$$

Now let's zoom in again between 1.4 and 1.5:

$$\begin{array}{lcll} 1.41^2 & = & 1.9881 & \text{slightly below 2} \\ 1.42^2 & = & 2.0164 & \text{slightly above 2} \end{array}$$

Hence,

$$1.41 < \sqrt{2} < 1.42.$$

Trying one more digit:

$$1.414^2 = 1.999396, \qquad 1.415^2 = 2.002225.$$

So,

$$1.414 < \sqrt{2} < 1.415,$$

which gives the familiar approximation

$$\boxed{\sqrt{2} \approx 1.4142}.$$

Fun fact: The value $1.41421356\ldots$ has been known since ancient times — the Babylonians approximated it as

$$\frac{99}{70} \approx 1.4142857,$$

accurate to within one part in 10,000! Their clay tablets show that even 3,800 years ago, people had clever numerical methods to estimate square roots.

Quick Check

Without a calculator, estimate each value.

1. $\sqrt{5}$ (between which two whole numbers?)
2. $\sqrt{20}$ (between which two whole numbers?)

Square Roots and Exponents

We already know how to raise numbers to integer powers:

$$a^n = a \times a \times \cdots \times a.$$

We are also aware of some properties of exponents, such as $a^m \cdot a^n = a^{m+n}$.

But what about fractional exponents? Let's experiment:

$$4^{\frac{1}{2}} \times 4^{\frac{1}{2}} = 4^{\frac{1}{2}+\frac{1}{2}} = 4^1 = 4.$$

So $4^{\frac{1}{2}}$ must be the number that multiplied by itself gives 4 — i.e. the square root of 4, i.e. 2.

Power $\frac{1}{2}$

$$\boxed{a^{\frac{1}{2}} = \sqrt{a}}$$

This keeps all the exponent rules consistent and allows us to define fractional powers with denominator $= 2$:

$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a, \qquad (a^m)^{\frac{1}{2}} = a^{\frac{m}{2}}.$$

Beyond Square Roots

The idea of taking a root is not limited to squares.

The **cube root** of a number a is a number whose cube (third power) equals a:

$$x^3 = a \quad \Longleftrightarrow \quad x = \sqrt[3]{a}.$$

For example, $\sqrt[3]{8} = 2$ because $2^3 = 8$, and $\sqrt[3]{-27} = -3$ because $(-3)^3 = -27$.

Cube roots can be positive or negative, since a negative number cubed remains negative:

$$\sqrt[3]{8} = 2, \quad \sqrt[3]{-8} = -2.$$

Similarly, we can define the **fourth root**, **fifth root**, and so on:

$$\sqrt[n]{a} \text{ is the number whose } n\text{-th power equals } a.$$

That is,

$$x^n = a \quad \Longleftrightarrow \quad x = \sqrt[n]{a}.$$

$$\sqrt[4]{16} = 2 \text{ because } 2^4 = 16, \qquad \sqrt[5]{32} = 2 \text{ because } 2^5 = 32.$$

Roots and fractional powers

$$a^{\frac{1}{n}} = \sqrt[n]{a}, \qquad (a^{\frac{1}{n}})^n = a.$$

Fractional exponents and roots are two ways of expressing the same idea.

As we take higher and higher roots, all numbers move closer to 1:

- If $a > 1$, higher roots become **smaller** (closer to 1).
- If $0 < a < 1$, higher roots become **larger** (closer to 1).

For $a = 4$:

$$\sqrt{4} = 2, \qquad \sqrt[3]{4} = 1.587, \qquad \sqrt[4]{4} = 1.414, \qquad \sqrt[10]{4} = 1.149.$$

All approach 1 as the root's degree increases.
For $a = \frac{1}{4}$:

$$\sqrt{\tfrac{1}{4}} = \tfrac{1}{2} = 0.5, \qquad \sqrt[3]{\tfrac{1}{4}} = 0.63, \qquad \sqrt[4]{\tfrac{1}{4}} = 0.71.$$

Here the values rise toward 1.

Quick Check

Evaluate or explain.

1. $\sqrt[3]{-64}$
2. $\sqrt[4]{16}$
3. Why does $\sqrt[3]{-8}$ exist, but $\sqrt{-8}$ does not?

How Calculators Find Square Roots

When you press the $\sqrt{\ }$ button, your calculator does not actually "know" the answer. Instead, it *computes* it by solving the equation

$$x^2 = a,$$

using a fast method that dates back almost 4,000 years!

The Babylonian (Newton) Method

Start with a guess x_0 for $\sqrt{a}$, then repeatedly apply the rule

$$\boxed{x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)}.$$

Each new value x_{n+1} is a better approximation of $\sqrt{a}$.

Let's find $\sqrt{2}$ step by step, this time using *fractions* instead of decimals.
We start with $x_0 = 1$ and repeatedly apply

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right).$$

$$x_0 = 1$$

$$x_1 = \frac{1}{2}\left(1 + \frac{2}{1}\right) = \frac{3}{2} = 1.5$$

$$x_2 = \frac{1}{2}\left(\frac{3}{2} + \frac{2}{\frac{3}{2}}\right) = \frac{1}{2}\left(\frac{3}{2} + \frac{4}{3}\right) = \frac{1}{2}\left(\frac{17}{6}\right) = \frac{17}{12} = 1.4166\ldots$$

$$x_3 = \frac{1}{2}\left(\frac{17}{12} + \frac{2}{\frac{17}{12}}\right) = \frac{1}{2}\left(\frac{17}{12} + \frac{24}{17}\right) = \frac{1}{2}\left(\frac{577}{204}\right) = \frac{577}{408} \approx 1.41422$$

Already after three steps we get the answer correct to four decimal places! Each step rapidly reduces the error and brings us closer to the true value of $\sqrt{2}$.

Your computer or calculator performs the same kind of iteration, but in binary arithmetic and with high precision (often 16 digits or more). Within just a few rounds, it gets the exact value your screen shows.

The same idea works for other roots!

To find the r-th root of a (for example, cube root when $r = 3$), the computer uses:

$$x_{n+1} = \frac{1}{r}\left[(r-1)x_n + \frac{a}{x_n^{r-1}}\right].$$

For $\sqrt[3]{8}$:

$$x_0 = 1, \quad x_1 = \tfrac{1}{3}(2 \cdot 1 + 8/1^2) = 3.33, \quad x_2 = \tfrac{1}{3}(2 \cdot 3.33 + 8/3.33^2) = 2.15, \quad x_3 = 2.02.$$

In just a few steps, we reach 2!

Historical Note

The same method appears on ancient Babylonian clay tablets (about 1800 BCE). One tablet (YBC 7289) gives

$$\sqrt{2} \approx 1.414213,$$

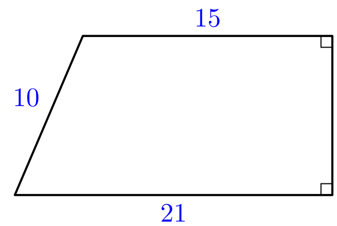
which matches modern calculators to six decimal places! Isaac Newton later rediscovered the same principle in the 1600s, and it remains the foundation of modern numerical algorithms.

Summary

- Rational numbers: can be written as $\frac{p}{q}$; decimal is finite or repeating.
- Irrational numbers: cannot be written as $\frac{p}{q}$; decimal is infinite and nonrepeating.
- $\sqrt{a}$ is the nonnegative number whose square equals a.
- $\sqrt{a} = a^{1/2}$ — fractional exponents are another way to write roots.
- Famous irrational numbers: $\sqrt{2}$, π, $\varphi = \frac{1+\sqrt{5}}{2}$ (the golden ratio).

Homework

1. Convert each fraction into a decimal:
   (a) $\frac{1}{8}$ (b) $\frac{5}{6}$ (c) $\frac{1}{13}$
2. Convert the following decimals into fractions:
   (a) 0.015 (b) 0.28 (c) $0.\overline{420}$ (d) $0.12\overline{5}$ (e) $0.6\overline{776}$
3. Convert the following fractions into decimals:
   $$\tfrac{1}{7},\quad \tfrac{2}{7},\quad \tfrac{3}{7},\quad \tfrac{4}{7},\quad \tfrac{5}{7},\quad \tfrac{6}{7}$$
   Do you notice any pattern in the decimal expansions?
4. Evaluate:
   $$\frac{10^2+11^2+12^2+13^2+14^2}{365}$$
5. Prove that each value is rational.
   (a) $(\sqrt{2}-1)(\sqrt{2}+1)$
   (b) $(\sqrt{5}-\sqrt{3})(\sqrt{5}+\sqrt{3})$
   (c) $(\sqrt{2}+1)^2+(\sqrt{2}-1)^2$
   (d) $(\sqrt{7}-1)^2+(\sqrt{7}+1)^2$
   (e) $(\sqrt{7}-2)^2+4\sqrt{7}$
6. Without a calculator, compare each pair (write $<$, $>$, or $=$).
   $$3 \square \sqrt{11} \qquad 11 \square \sqrt{110} \qquad 22 \square \sqrt{484}$$
   $$5 \square \sqrt{20} \qquad 17 \square \sqrt{299} \qquad 35 \square \sqrt{1215}$$
7. In each right triangle below, find the length of the unknown side.
   (a) The legs measure 2 and 5. Find the hypotenuse.
   (b) The hypotenuse measures 7, and one leg is 6. Find the other leg.
8. Evaluate (simplify radicals when possible).
   (a) $5\cdot\sqrt{4}\cdot 3$
   (b) $2\cdot\sqrt{9}+3\cdot\sqrt{16}$
   (c) $\sqrt{13-3\cdot 3}$
   (d) $\sqrt{7^2-\frac{26}{2}}$
   (e) $\frac{1}{2}\sqrt{5^2+\frac{22}{2}}$
   (f) $3\sqrt{0.64}-5\sqrt{1.21}$
9. Find the following square roots. If you cannot find the exact number, state between which two whole numbers it lies (for example, "between 5 and 6").
   (a) $\sqrt{16}$ (b) $\sqrt{81}$ (c) $\sqrt{10{,}000}$ (d) $\sqrt{10^8}$ (e) $\sqrt{50}$
10. Simplify each square root using the identity $\sqrt{ab}=\sqrt{a}\sqrt{b}, \qquad a,b\geq 0.$
    (a) $\sqrt{18}$ (b) $\sqrt{27}$ (c) $\sqrt{32}$
11. **Find:**
    $$\sqrt{2^6\times 7^2}, \qquad \sqrt{\frac{1}{16}}, \qquad \sqrt{\frac{4}{9}}.$$
12. Find the height and area of the figure below. Lengths of three sides are given; the two marked angles are right angles.



13. The side of an equilateral triangle is 1 m. Find its height and its area.

*14. **Iterative experiment.** Take some positive number $x < 100$ and, using a calculator or computer, compute

$$x_{\text{new}} = \frac{x}{2} + \frac{1}{x}.$$

Then repeat the same calculation 10 times, each time using the previous result as the new x. Finally, square the result.

What number do you get? Try the same experiment starting with different initial values of x. Is the result surprising?