MATH 5: HANDOUT 12

Fractions and Decimals

Fractions Refresher

Prime Factorization, GCD, and LCM

- A number a is a **divisor** (or **factor**) of a number b if it divides b exactly, that is, if $b \div a$ is an integer.

  Example: 3 is a divisor of 12 because $12 \div 3 = 4$.

  $$a \mid b \quad \text{means “}a\text{ divides }b\text{.”}$$

- A number b is a **multiple** of a if it can be written as $b = a \times k$ for some integer k.

  Example: 12 is a multiple of 3 since $12 = 3 \times 4$.

- A **prime number** is a number greater than 1 that has no divisors other than 1 and itself. Alternatively, we can say that a prime number is a number with exactly two divisors — that will exclude 1 automatically. Examples: $2, 3, 5, 7, 11, 13, 17, \ldots$

- Every integer greater than 1 can be written uniquely as a product of prime numbers. This is called its **prime factorization**.

  $$84 = 2^2 \cdot 3 \cdot 7$$

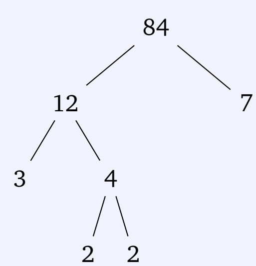
Finding Prime Factorization

There are two main ways to find the prime factorization of a number.

1. Repeated Division Method:
Keep dividing the number by the smallest possible prime until you reach 1.

$$\begin{array}{r|l} 84 & 2 \\ 42 & 2 \\ 21 & 3 \\ 7 & 7 \\ 1 & \text{stop} \end{array} \qquad \Rightarrow \qquad 84 = 2 \times 2 \times 3 \times 7 = 2^2 \cdot 3 \cdot 7.$$

2. Factor Tree Method:
Split the number into any two factors and keep breaking down until all factors are prime.



Again, $84 = 2^2 \cdot 3 \cdot 7$.

Both methods give the same result. The order of the factors may differ, but the set of primes and their powers is always unique!

Greatest Common Divisor (GCD): The *GCD* (also called the *greatest common factor*, GCF) of two or more numbers is the largest number that divides all of them.

Methods:

1. **By listing factors:**
$$12: \{1,2,3,4,6,12\}, \quad 18: \{1,2,3,6,9,18\}.$$
Common factors: $\{1,2,3,6\}$. So $\mathrm{GCD}(12,18)=6$.
2. **By prime factorization:**
$$12 = 2^2 \cdot 3, \quad 18 = 2 \cdot 3^2.$$
Take the smallest powers of common primes: $2^1 \cdot 3^1 = 6$.

Least Common Multiple (LCM): The *LCM* of two or more numbers is the smallest positive number that is a multiple of all of them.

Methods:

1. **By listing multiples:**
$$12: \ 12, 24, 36, 48, 60, \dots \quad 18: \ 18, 36, 54, 72, \dots$$
The first common multiple is 36, so $\mathrm{LCM}(12,18)=36$.
2. **By prime factorization:**
$$12 = 2^2 \cdot 3, \quad 18 = 2 \cdot 3^2.$$
Take the largest powers of each prime: $2^2 \cdot 3^2 = 36$.

Connection between GCD and LCM: For any positive integers a and b,
$$a \cdot b = \mathrm{GCD}(a,b) \cdot \mathrm{LCM}(a,b).$$

These ideas — prime factorization, GCD, and LCM — form the foundation for working with rational numbers, and we'll use them again when we discuss decimals, reciprocals, and sets of numbers.

Quick Check

1. Find the prime factorization of 90.
2. Find $\gcd(18,30)$ using prime factorizations.
3. Find $\mathrm{lcm}(8,12)$ using prime factorizations.
4. Check that the relationship
$$a \cdot b = \gcd(a,b) \cdot \mathrm{lcm}(a,b)$$
holds for $a = 15$ and $b = 20$.

Fractions: Review of Operations

A **fraction** represents a part of a whole. It has two parts:
$$\frac{\text{numerator}}{\text{denominator}}.$$

- The **numerator** (top number) tells how many parts we have.
- The **denominator** (bottom number) tells how many equal parts the whole is divided into.

Examples:
$$\tfrac{1}{2} \text{ means 1 part out of 2 equal parts (a half)}, \quad \tfrac{3}{4} \text{ means 3 parts out of 4 equal parts.}$$

Proper and Improper Fractions:

- A **proper fraction** has a numerator smaller than the denominator, e.g. $\frac{3}{5}$.
- An **improper fraction** has a numerator equal to or larger than the denominator, e.g. $\frac{7}{5}$.

Equivalent Fractions: Two fractions are **equivalent** if they represent the same part of a whole, even if the numbers look different.

$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{50}{100}.$$

To find an equivalent fraction, multiply or divide both numerator and denominator by the same nonzero number:

$$\frac{2}{3} = \frac{2\times 5}{3\times 5} = \frac{10}{15}.$$

The value doesn't change, only the way it's written.

Simplifying (Reducing) Fractions: To simplify a fraction, divide numerator and denominator by their greatest common divisor (GCD):

$$\frac{24}{36} = \frac{24\div 12}{36\div 12} = \frac{2}{3}.$$

Remember:

Changing how a fraction looks does not change its value.

Equivalent fractions are different names for the same number.

Finding Common Denominators: When adding or comparing fractions, rewrite them with a common denominator — usually the LCM of denominators.

Adding and Subtracting Fractions:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd},$$
$$\frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}.$$

Examples:

$$\frac{2}{3} + \frac{5}{4} = \frac{8+15}{12} = \frac{23}{12} = 1\frac{11}{12}, \quad \frac{3}{5} - \frac{1}{10} = \frac{6-1}{10} = \frac{1}{2}.$$

Multiplying and Dividing Fractions:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}, \quad \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{ad}{bc}.$$

Example:

$$\frac{2}{3} \times \frac{9}{10} = \frac{18}{30} = \frac{3}{5}, \quad \frac{5}{8} \div \frac{3}{4} = \frac{5}{8} \times \frac{4}{3} = \frac{5}{6}.$$

Mixed Numbers and Improper Fractions A **mixed number** combines a whole number and a proper fraction:

$$1\frac{3}{4}, \quad 2\frac{2}{5}, \quad 5\frac{7}{8}.$$

An **improper fraction** is one where the numerator is greater than or equal to the denominator:

$$\frac{7}{4}, \quad \frac{12}{5}, \quad \frac{47}{8}.$$

Conversion:

$$1\frac{3}{4} = \frac{1\cdot 4+3}{4} = \frac{7}{4},$$
$$\frac{11}{5} = 2\frac{1}{5} \quad (\text{since } 11 = 2\cdot 5 + 1).$$

When to Use Each Form:

- Use **mixed numbers** when describing quantities or measurements in words (e.g., “$2\frac{1}{2}$ hours” or “$3\frac{3}{4}$ cups of flour”). They’re easier to read and visualize.
- Use **improper fractions** when performing calculations — addition, subtraction, multiplication, or division — because they are easier to work with algebraically.

Example:

$$1\tfrac{3}{4} + 2\tfrac{2}{3} = \tfrac{7}{4} + \tfrac{8}{3} = \tfrac{21+32}{12} = \tfrac{53}{12} = 4\tfrac{5}{12}.$$

We convert to improper fractions to compute, and then (optionally) back to a mixed number at the end.

Quick Check

1. Decide whether each fraction is proper or improper:
$$\frac{7}{10}, \quad \frac{15}{8}, \quad \frac{9}{9}.$$
2. Simplify the fraction $\dfrac{18}{30}$.
3. Compute and simplify:
$$\frac{3}{4} + \frac{5}{6}.$$
4. Compute and simplify:
$$\frac{7}{9} \cdot \frac{6}{7}.$$
5. Rewrite $2\dfrac{3}{5}$ as an improper fraction, then subtract:
$$2\frac{3}{5} - \frac{4}{5}.$$

Sets

A *set* is a collection of elements. In mathematics, sets usually contain numbers. Although the idea is simple, giving a fully precise definition of a “set” is actually quite difficult, and mathematicians usually work with examples and intuitive understanding rather than formal definitions at this level.

The main sets of numbers we work with are:

- $\mathbb{N}$: Natural numbers $1, 2, 3, \ldots$.
  - Operations that stay inside $\mathbb{N}$: addition ($2 + 3 = 5$), multiplication ($2 \cdot 3 = 6$).
  - But subtraction does not always stay in $\mathbb{N}$: for example, $1 - 2$ is not a natural number. To handle such cases, we need a larger set.
- $\mathbb{Z}$: Integers $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$.
  - Here we can add, subtract, and multiply without leaving the set of integers.
  - But division is a problem: $\frac{1}{2}$ is not an integer. To include such numbers, we need another expansion.
- $\mathbb{Q}$: Rational numbers, i.e., numbers that can be written as fractions $\frac{p}{q}$ where $p, q \in \mathbb{Z}$ and $q \neq 0$.
  - In $\mathbb{Q}$ we can add, subtract, multiply, and divide (except division by 0).
  - This makes $\mathbb{Q}$ the most flexible set for arithmetic.

Thus we have a natural hierarchy:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q}.$$

Each new set of numbers was created because the previous one was not sufficient to perform all the operations we wanted.

Example: $5 \in \mathbb{N}$, $-7 \in \mathbb{Z}$, $\frac{3}{4} \in \mathbb{Q}$.

Quick Check

1. For each number, say which sets it belongs to:

$$4, \quad -3, \quad \frac{5}{2}.$$

   (Choose from $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$; some numbers belong to more than one.)

2. Give an example of a subtraction problem with natural numbers whose difference is *not* a natural number.
3. Give an example of a division problem with integers whose quotient is *not* an integer but is a rational number.
4. Place these sets in order using $\subset$ symbols:

$$\mathbb{N}, \ \mathbb{Q}, \ \mathbb{Z}.$$

Fractions and Decimals

Every rational number can be expressed as a decimal. Some fractions give finite decimals (e.g., $\frac{1}{2} = 0.5$), others give infinite repeating decimals (e.g., $\frac{2}{7} = 0.285714285714\cdots = 0.\overline{285714}$).

We convert fractions to decimals using long division. The process either ends (finite decimal) or eventually repeats (repeating decimal).

Example 1 (Finite (or terminating) decimal):

$$\frac{3}{4} = 0.75$$

Example 2 (Repeating decimal): Perform long division for $\frac{2}{7}$:

	0	.	2	8	5	7	1	4	2
7	2	.	0	0	0	0	0	0	0
	1	.	4						
			6	0					
		−	5	6					
				4	0				
			−	3	5				
					5	0			
				−	4	9			
						1	0		
					−		7		
							3	0	
						−	2	8	
								2	0

$$0.285714285714\cdots = 0.\overline{285714}.$$

Why Every Fraction Either Ends or Repeats

When we convert a fraction to a decimal using long division, only two things can happen:

- The division **ends** (remainder becomes 0) — giving a *finite (terminating) decimal*.
- The division **never ends**, but the same remainders start to *repeat* — giving a *repeating decimal*.

Why must one of these happen?
When dividing by a denominator d, each remainder must be one of the numbers

$$0, \ 1, \ 2, \ 3, \ \ldots, \ d-1.$$

There are only d possible remainders! If the remainder becomes 0, the decimal ends. If not, some remainder must eventually repeat — and from that point on, the digits will repeat too.

Examples:

$$\tfrac{1}{2} = 0.5 \quad \text{(ends because remainder} = 0)$$
$$\tfrac{1}{3} = 0.333\ldots = 0.\overline{3} \quad \text{(repeats because remainder repeats)}$$
$$\tfrac{2}{7} = 0.285714285714\ldots = 0.\overline{285714}$$

So *every rational number (fraction)* gives either a terminating or repeating decimal—never something else. Numbers whose decimals never end and never repeat (like π or $\sqrt{2}$) are called **irrational**.

Which Fractions End and Which Repeat? A fraction in simplest form $\frac{p}{q}$ will have a **terminating decimal** if and only if the denominator q has no prime factors other than 2 or 5.

If q contains any other prime factor (like 3, 7, or 11), the decimal will be **repeating**.

Fraction	Denominator factors	Decimal type
$1/2$	2	terminating (0.5)
$1/4$	2^2	terminating (0.25)
$1/5$	5	terminating (0.2)
$1/8$	2^3	terminating (0.125)
$1/3$	3	repeating ($0.\overline{3}$)
$1/6$	2×3	repeating ($0.1\overline{6}$)
$1/7$	7	repeating ($0.\overline{142857}$)
$1/9$	3^2	repeating ($0.\overline{1}$)
$1/12$	$2^2 \times 3$	repeating ($0.08\overline{3}$)

Rule to remember:

Only denominators of the form $2^m 5^n$ give finite decimals.

All others produce repeating decimals.

Why Only Denominators with 2s and 5s Give Finite Decimals

Every decimal comes from dividing a numerator by a denominator. A decimal stops (terminates) only if the denominator "fits" into some power of 10.

Step 1. Powers of 10 look like this:

$$10 = 2 \times 5, \qquad 100 = 10^2 = (2 \times 5)^2 = 2^2 \times 5^2, \qquad 1000 = 10^3 = (2 \times 5)^3 = 2^3 \times 5^3, \text{ and so on.}$$

Every power of 10 has only the prime factors 2 and 5.

Step 2. If the denominator of a fraction (after simplifying) has only 2s and 5s, we can multiply top and bottom by something to make the denominator a power of 10. Then the fraction will become an exact decimal.

$$\frac{3}{8} = \frac{3}{2^3} = \frac{3}{2^3} \times \frac{5^3}{5^3} = \frac{3 \times 125}{2^3 \times 5^3} = \frac{3 \times 125}{1000} = \frac{375}{1000} = 0.375$$
$$\frac{7}{20} = \frac{7}{2^2 \times 5} = \frac{7}{2^2 \times 5} \times \frac{5}{5} = \frac{7 \times 5}{2^2 \times 5^2} = \frac{35}{100} = 0.35$$

Step 3. But if the denominator contains any other prime factor (like 3, 7, or 11), no power of 10 will divide evenly by that factor, so the division will never end — the remainder will eventually repeat.

$$\frac{1}{6} = \frac{1}{2 \times 3} \text{ has a 3 in the denominator} \Rightarrow 0.1\overline{6}.$$

$$\frac{1}{7} \text{ has a } 7 \Rightarrow 0.\overline{142857}.$$

Conclusion:

If the denominator has only 2s and 5s, it can be turned into a power of 10, so the decimal ends.

All other denominators give repeating decimals.

Quick Check

1. Decide whether each fraction has a terminating or repeating decimal. Justify using the prime factorization of the denominator.
$$\frac{7}{20}, \quad \frac{5}{12}, \quad \frac{9}{25}.$$
2. Without doing full long division, explain why $\frac{1}{40}$ has a finite decimal.
3. Without doing full long division, explain why $\frac{3}{14}$ has a repeating decimal.

Converting Decimals to Fractions

Every decimal number represents a fraction whose denominator is a power of 10 — possibly simplified later.

1. Finite (terminating) decimals Move the decimal point to make an integer and divide by the corresponding power of 10 (1 followed by a suitable number of zeroes).

$$0.6 = \tfrac{6}{10} = \tfrac{3}{5}, \qquad 0.125 = \tfrac{125}{1000} = \tfrac{1}{8}, \qquad 3.75 = \tfrac{375}{100} = \tfrac{15}{4}.$$

Rule: If a decimal has n digits after the decimal point, multiply numerator and denominator by 10^n (1 followed by n zeroes) to remove the decimal, then simplify.

$$\text{Example: } 0.072 = \tfrac{72}{1000} = \tfrac{9}{125}.$$

2. Repeating decimals Let the repeating block be called the *repetend*. Use algebra to find the fraction.
Example 1: Convert $0.\overline{3}$ to a fraction.

$$x = 0.3333\ldots$$

Multiply both sides by 10:

$$10x = 3.3333\ldots$$

Subtract the first equality:

$$9x = 3 \quad \Rightarrow \quad x = \tfrac{1}{3}.$$

Example 2: Convert $0.\overline{27}$ to a fraction.

$$x = 0.272727\ldots$$

Multiply by 100 (since 2 digits repeat):

$$100x = 27.272727\ldots$$

Subtract the first equality:

$$99x = 27 \quad \Rightarrow \quad x = \tfrac{27}{99} = \tfrac{3}{11}.$$

Example 3: Convert $0.1\overline{6}$ to a fraction.

$$x = 0.1666\ldots$$

Multiply by 10:

$$10x = 1.6666\ldots$$

Multiply by 100:

$$100x = 16.6666\ldots$$

Subtract the first from the second:

$$90x = 15 \quad \Rightarrow \quad x = \tfrac{15}{90} = \tfrac{1}{6}.$$

Repeating decimals always correspond to rational numbers — this is one reason why we call all fractions and repeating decimals "rational numbers."

Why $0.\overline{9} = 1$

At first glance, $0.9999\ldots$ looks like it should be just a tiny bit less than 1. However, mathematically they are exactly equal.

Algebraic reasoning:
Let

$$x = 0.9999\ldots$$

Multiply both sides by 10:

$$10x = 9.9999\ldots$$

Now subtract the first equality from the second:

$$10x - x = 9.9999\ldots - 0.9999\ldots = 9$$

so

$$9x = 9 \quad \Rightarrow \quad x = 1.$$

Hence, $0.9999\ldots = 1$.

Conclusion:

$0.\overline{9}$ and 1 are two different ways to write the same number.

The number line has no "gap" between them.

Irrational Numbers and Why $\sqrt{2}$ Is Not Rational

Not every number can be written as a fraction!
Rational numbers are those that can be expressed as $\frac{p}{q}$, where p and q are integers and $q \neq 0$.
Irrational numbers are numbers that *cannot* be written as a fraction. Their decimal expansions never end and never repeat.

Examples:

$$\pi = 3.1415926535\ldots, \qquad \sqrt{2} = 1.414213562\ldots$$

These decimals go on forever with no pattern.

A simple proof that $\sqrt{2}$ is not rational:
Suppose, just for the sake of argument, that $\sqrt{2}$ *is* rational. Then we could write it as a fraction in lowest terms:

$$\sqrt{2} = \tfrac{a}{b},$$

where a and b are integers that share no common factor.

Now square both sides:

$$2 = \tfrac{a^2}{b^2} \quad \Rightarrow \quad a^2 = 2b^2.$$

This means a^2 is even (because it's 2 times another integer), so a must also be even. Let's write $a = 2k$.

Substitute back:

$$(2k)^2 = 2b^2 \quad \Rightarrow \quad 4k^2 = 2b^2 \quad \Rightarrow \quad b^2 = 2k^2.$$

Now b^2 is also even, so b is even too.

But then both a and b are even — they have a common factor 2. That contradicts our assumption that the fraction was in simplest form.

$\Rightarrow$ Our assumption was wrong: $\sqrt{2}$ cannot be written as a fraction.

Conclusion:

$\sqrt{2}$ and π are examples of irrational numbers — their decimals never end and never repeat.

Together with the rational numbers, they form the set of **real numbers** $\mathbb{R}$:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}.$$

Quick Check

1. Write each finite decimal as a fraction in simplest form:

$$0.48, \quad 2.05, \quad 0.007.$$

2. Convert $0.\overline{6}$ to a fraction using algebra.
3. Convert $0.2\overline{7}$ to a fraction using algebra.
4. True or false: every repeating decimal represents a rational number. Explain briefly.

Reciprocals

Definition. For any nonzero number a, the *reciprocal* of a is the number that multiplies with a to give 1.

$$a \cdot r(a) = 1.$$

We often write the reciprocal of a as $\frac{1}{a}$.

$$r(a) = \tfrac{1}{a}, \qquad r\left(\tfrac{p}{q}\right) = \tfrac{q}{p}.$$

Examples:

$$r(5) = \tfrac{1}{5}, \quad r\left(\tfrac{3}{8}\right) = \tfrac{8}{3}, \quad r(0.2) = 5, \quad r\left(1\tfrac{1}{3}\right) = \tfrac{3}{4}.$$

Why do we need reciprocals? Reciprocals let us "undo" multiplication — just like negative numbers let us "undo" addition.

$$\begin{array}{ll} \text{Additive opposite:} & a + (-a) = 0 \\ \text{Multiplicative opposite:} & a \times \frac{1}{a} = 1 \end{array}$$

So: - The number $-a$ cancels a when we add. - The number $\frac{1}{a}$ cancels a when we multiply.

Example:

$$5 \times \tfrac{1}{5} = 1, \quad \tfrac{3}{4} \times \tfrac{4}{3} = 1, \quad (-2) \times \left(-\tfrac{1}{2}\right) = 1.$$

Notice that the reciprocal of a negative number is also negative.

Why reciprocals matter.

- **Division:** Dividing by a number means multiplying by its reciprocal:

$$\tfrac{a}{b} \div \tfrac{c}{d} = \tfrac{a}{b} \times \tfrac{d}{c}.$$

- **Solving equations:** To "get rid of" a coefficient in multiplication, multiply both sides by its reciprocal.

$$5x = 20 \quad \Rightarrow \quad x = 20 \times \tfrac{1}{5} = 4.$$

Note: Zero has no reciprocal, because no number multiplied by 0 can ever give 1.

Opposites and Reciprocals

In math, every operation has a way to "undo" it.

Concept	Operation it undoes	Example
Additive opposite (negative)	Addition	$5 + (-5) = 0$
Multiplicative opposite (reciprocal)	Multiplication	$5 \times \frac{1}{5} = 1$

So:

To "cancel" a number under addition, use its negative.

To "cancel" a number under multiplication, use its reciprocal.

Examples:

$$-7 \text{ is the additive opposite of } 7, \quad 7 + (-7) = 0.$$
$$\tfrac{1}{7} \text{ is the multiplicative opposite of } 7, \quad 7 \times \tfrac{1}{7} = 1.$$

Just like 0 is the "neutral" number for addition, 1 **is the neutral number for multiplication.**

$$a + 0 = a, \qquad a \times 1 = a.$$

Quick Check

1. Find the reciprocal of each number:

$$4, \quad -\frac{3}{5}, \quad 0.25.$$

2. Which number has no reciprocal? Explain why.

3. Use reciprocals to solve the equation

$$\frac{3}{4}x = 9.$$

4. Check that each pair of numbers are reciprocals by multiplying them:

$$\frac{5}{6} \text{ and } \frac{6}{5}, \qquad -2 \text{ and } -\frac{1}{2}.$$