MATH 7: HANDOUT 11

Numeral Systems

Numeral Systems Through History

Over the long centuries of human history, many different **numeral systems** have appeared in different cultures. The oldest systems were not *place-valued*: each symbol had its own meaning, and the value of a number did not depend on its position. Numbers were written by repeating symbols and adding their values.

Ancient Egyptian System

One of the best-known examples of such a system is the **ancient Egyptian decimal system**. It was based on powers of 10, just like ours, but it had no zero and no positional place values. Symbols could appear in different orders and were often read from right to left or from bottom to top.

The Egyptian hieroglyphs given on the right, from the Temple of Edfu (237–57 BCE), represent the number 1,333,330:

𓁨 𓆐 𓆐 𓆐 𓂭 𓂭 𓂭 𓆼 𓆼 𓆼 𓍢 𓍢 𓍢 𓎆 𓎆 𓎆

𓁨𓆐𓆐𓆐𓂭𓂭𓂭𓆼𓆼𓆼𓍢𓍢𓍢𓎆𓎆𓎆

Temple of Edfu inscription
(History of Ancient Egyptian Numbers)

The Egyptian symbols for powers of ten are:

- **Single stroke** (𓏺) —represents 1
- **Heel bone** (𓎆) —represents 10
- **Scroll or rope coil** (𓍢) —represents 100
- **Lotus flower** (𓆼) —represents 1,000
- **Pointing finger** (𓂭) —represents 10,000
- **Tadpole or frog** (𓆐) —represents 100,000
- **Astonished god (Heh)** (𓁨) —represents 1,000,000

1	10	100	1,000	10,000	100,000	1,000,000
𓏺	𓎆	𓍢	𓆼	𓂭	𓆐	𓁨

To write a number, each symbol is repeated up to nine times.
For example:

4,321 = 𓆼 𓆼 𓆼 𓆼 𓍢 𓍢 𓍢 𓎆 𓎆 𓏺 = 𓆼𓆼𓆼𓆼𓍢𓍢𓍢𓎆𓎆𓏺

57,213 = 𓂭 𓂭 𓂭 𓂭 𓂭 𓆼 𓆼 𓆼 𓆼 𓆼 𓆼 𓆼 𓍢 𓍢 𓎆 𓏺 𓏺 𓏺 = 𓂭𓂭𓂭𓂭𓂭𓆼𓆼𓆼𓆼𓆼𓆼𓆼𓍢𓍢𓎆𓏺𓏺𓏺

In other words: Egyptians simply "stacked" symbols and added them. There was no special placeholder for "nothing in the tens place," because there were no places. This is very different from how we write, for example, 5 0 3 to mean "five hundreds, zero tens, three ones."

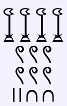
An Egyptian Inscription from the Louvre

The inscription on the right (from the Louvre Museum) shows numerals carved in columns. The calf at the top faces to the right, so the text is read from top to bottom and, within each line, from right to left.

The number shown consists of:

- four lotus flowers ($4 \times 1{,}000$),
- six coils of rope (6×100),
- two hobbles for cattle (2×10),
- and two single strokes (2×1).

Altogether this is $4{,}622$. Since this inscription comes right after the word "calf," it means "4,622 calves."

4,622 =  = 

Inscription from the Louvre (An Ancient Egyptian Mathematical Photo Album)

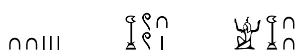
Practice: Egyptian Numerals

1. Write the following numbers using Egyptian symbols:

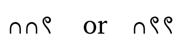
   $$37, \quad 205, \quad 1{,}204, \quad 6{,}315.$$

   (Hint: use the correct number of strokes, heel bones, coils, and lotus flowers.)

2. Write the following Egyptian numerals as regular numbers:

3. Which of these two numbers is larger?

Fractions. Fractions in ancient Egypt were almost exclusively unit fractions. The notation that was used to signify a fraction: a mouth hieroglyph representing "part." The rare exceptions to unit fractions include special symbols for $\frac{1}{2}$, $\frac{2}{3}$, $\frac{1}{4}$, and $\frac{3}{4}$.

Egyptian fractions: $\frac{1}{6}$, $\frac{1}{16}$, $\frac{1}{120}$, $\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$, $\frac{2}{3}+\frac{1}{6}=\frac{5}{6}$. In the right two pictures, notice special symbols for $\frac{1}{2}$ and $\frac{2}{3}$. (An Ancient Egyptian Mathematical Photo Album)

Babylonian System

Another ancient civilization, the **Babylonians** (around 2000 BCE), invented a completely different system. The Babylonians used only two wedge-shaped symbols to write every number from 1 to 59. Their numeration was based on 60 (a **sexagesimal** system). Inside each "digit," they combined groups of 10-wedges and 1-wedges, so it still *looks* a little bit like tens and ones.

Even more importantly, the Babylonian system was **positional**. The same cluster of wedges could mean "12," or "12 sixties," or "12 sixty-squared," depending on where it appeared. For example, the number 62 was written as

𒁹 𒈫

which means $1 \times 60 + 2$.

Base-60 in the Modern World

The Babylonian system may be ancient, but traces of it are still all around us!

- **Time:** We divide an hour into **60 minutes**, and each minute into **60 seconds**. This idea goes directly back to ancient Babylonian astronomy, where base-60 made it easy to divide circles and time intervals into many equal parts.
- **Angles and Circles:** A full circle has **360 degrees –** that' s 6×60. Each degree has **60 minutes** (′) and each minute has **60 seconds** (″). Again, this comes from Babylonian geometry, where the sky was imagined as a great 360-part circle.
- **Geographic Coordinates:** Latitude and longitude are written in degrees, minutes, and seconds:

  $$40°48'25'' \text{ N}, \quad 73°7'23'' \text{ W}.$$

  This is the same base-60 subdivision the Babylonians used for stars and planets.

Why 60? Sixty is a very convenient number for dividing:

$$60 = 2^2 \times 3 \times 5,$$

so it has **12 factors —** more than any smaller positive number. That means it divides evenly by 2, 3, 4, 5, and 6, making fractions neat:

$$\tfrac{60}{2} = 30, \quad \tfrac{60}{3} = 20, \quad \tfrac{60}{4} = 15, \quad \tfrac{60}{5} = 12, \quad \tfrac{60}{6} = 10.$$

No wonder the Babylonians liked it!

Babylonian numerals from 1 to 60:

1	𒁹	2	𒈫	3	𒐈	4	𒐉	5	𒐊
6	𒐋	7	𒑂	8	𒑄	9	𒑆	10	𒌋
11	𒌋𒁹	12	𒌋𒈫	13	𒌋𒐈	14	𒌋𒐉	15	𒌋𒐊
16	𒌋𒐋	17	𒌋𒑂	18	𒌋𒑄	19	𒌋𒑆	20	𒎙
21	𒎙𒁹	22	𒎙𒈫	23	𒎙𒐈	24	𒎙𒐉	25	𒎙𒐊
26	𒎙𒐋	27	𒎙𒑂	28	𒎙𒑄	29	𒎙𒑆	30	𒌍
31	𒌍𒁹	32	𒌍𒈫	33	𒌍𒐈	34	𒌍𒐉	35	𒌍𒐊
36	𒌍𒐋	37	𒌍𒑂	38	𒌍𒑄	39	𒌍𒑆	40	𒐏
41	𒐏𒁹	42	𒐏𒈫	43	𒐏𒐈	44	𒐏𒐉	45	𒐏𒐊
46	𒐏𒐋	47	𒐏𒑂	48	𒐏𒑄	49	𒐏𒑆	50	𒐐
51	𒐐𒁹	52	𒐐𒈫	53	𒐐𒐈	54	𒐐𒐉	55	𒐐𒐊
56	𒐐𒐋	57	𒐐𒑂	58	𒐐𒑄	59	𒐐𒑆	60	𒁹

Because Babylonian is positional, bigger numbers work like this:

$$𒐕\ 𒐖 \quad = \quad 1 \times 60 + 2 = 62$$

$$𒐖\ 𒐙 \quad = \quad 2 \times 60 + 5 = 125$$

$$𒐖\ 𒌍𒑂\ 𒌋𒐖 \quad = \quad 2 \times 60^2 + 37 \times 60 + 12 \quad = \quad 9{,}432$$

To show Babylonian place values using our digits, we separate the base–60 "digits' ' with commas. For example,

$$𒐕\ 𒐐𒑂\ 𒐏𒐚\ 𒐏 = (1, 57, 46, 40)_{60} \;=\; 1 \cdot 60^3 + 57 \cdot 60^2 + 46 \cdot 60 + 40 \;=\; 424{,}000_{10}.$$

So the Babylonian number "1,57,46,40" means 424,000 in our decimal notation.
Interestingly, Babylonians lacked a symbol for 0, and just left a space where 0 was supposed to be, so Babylonian number for 60 𒐕 was not different from their number for 1, or $60^2 = 3,600$.

Practice: Babylonian Numerals

1. Write the following base-10 numbers in the Babylonian base-60 system (showing each "digit" as a number between 0 and 59):

   $$73, \quad 125, \quad 3{,}600, \quad 3{,}726.$$

   (Hint: divide by 60 repeatedly.)

2. Express these Babylonian numbers in our base-10 system:

   $$𒐕\ 𒌋𒐗, \qquad 𒐖\ 𒐙\ 𒌍, \qquad 𒎙𒑄\ 𒐏𒐙.$$

   (Example: $𒐘\ 𒌋\ 𒎙𒐕 = (4, 10, 21)_{60} = 4 \times 60^2 + 10 \times 60 + 21 = 15{,}021.$)

Roman Numerals

Long after the Egyptians, the Romans used their own system of writing numbers. We still see it today on clocks, movie credits (MCMLXXXIV for 1984), book chapters, and the Super Bowl.

Roman numerals are built from just seven basic symbols:

Symbol	Value
I	1
V	5
X	10
L	50
C	100
D	500
M	1000

How Roman numerals are formed.

- If a symbol of *smaller or equal value* comes *after* a larger one, you **add**. Example: VI $= 5 + 1 = 6$. Example: XVIII $= 10 + 5 + 1 + 1 + 1 = 18$.
- If a symbol of *smaller value* comes *before* a larger one, you **subtract**. Example: IV $= 5 - 1 = 4$. Example: IX $= 10 - 1 = 9$.
- You write numbers by combining these additions and subtractions, biggest parts first.

Important subtraction patterns. The Romans did *not* just put anything before anything else. Only certain "subtract" pairs are allowed:

- I can go before V (5) or X (10): IV $= 4$, IX $= 9$.
- X can go before L (50) or C (100): XL $= 40$, XC $= 90$.
- C can go before D (500) or M (1000): CD $= 400$, CM $= 900$.

Other "creative" subtractions like IL for 49 or XM for 990 are **not standard**.

Reading Roman numerals. To read a Roman numeral:

1. Look left to right.
2. If a symbol is *at least as big* as the one after it, add it.
3. If it is *smaller* than the one after it, subtract it.

Example: MCMXLIV

$$\text{MCMXLIV} = \text{M} + (\text{CM}) + (\text{XL}) + (\text{IV}) = 1000 + 900 + 40 + 4 = 1944.$$

Let's check step by step:

- M $= 1000$
- CM $= 900$ $(1000 - 100)$
- XL $= 40$ $(50 - 10)$
- IV $= 4$ $(5 - 1)$

So MCMXLIV $= 1944$.

Writing Roman numerals. To *write* a number in Roman numerals, you usually:

1. Split the number by place values (thousands, hundreds, tens, ones).
2. Write each part using the allowed patterns.

Examples:

$$27 = 20 + 7 = \text{XX} + \text{VII} = \text{XXVII}.$$

$$49 = 40 + 9 = \text{XL} + \text{IX} = \text{XLIX}.$$

$$2024 = 2000 + 20 + 4 = \text{MM} + \text{XX} + \text{IV} = \text{MMXXIV}.$$

$$3999 = 3000 + 900 + 90 + 9 = \text{MMM} + \text{CM} + \text{XC} + \text{IX} = \text{MMMCMXCIX}.$$

Notice that:

- XX $= 20$ is just $10 + 10$.
- IX $= 9$ uses subtraction.
- MMM $= 3000$ repeats M up to three times.

Repetition rule. Some symbols can repeat up to three times in a row:

$$\text{III} = 3, \quad \text{XXX} = 30, \quad \text{CCC} = 300, \quad \text{MMM} = 3000.$$

But you cannot write four in a row. So:

$$4 \neq \text{IIII} \quad \text{(not allowed)}, \quad \text{use IV.}$$

$$40 \neq \text{XXXX}, \quad \text{use XL.}$$

Also: V, L, and D are never repeated twice in a row. There is no VV for 10, or LL for 100, or DD for 1000.

No zero. Roman numerals have symbols for 1, 5, 10, 50, ..., but there is **no symbol for** 0. This made certain arithmetic and bookkeeping tasks much harder than with the place-value system we use today. (Imagine trying to do long multiplication in Roman numerals. People basically didn't. They used counting boards / abaci.)

Practice: Roman Numerals

1. Write each in normal (Arabic) numbers:

$$\text{XLII}, \quad \text{CDXVI}, \quad \text{MMXVIII}, \quad \text{CMXC}.$$

2. Write each in Roman numerals:

$$14, \quad 70, \quad 944, \quad 2025.$$

Why we still care

Roman numerals are not very convenient for calculation —Egyptian numerals and Roman numerals are both "additive/subtractive" systems, not true place-value systems like ours. But they' re historically important, and they' re everywhere in art, archaeology, monuments, and old documents. Reading them is like reading a date carved in stone.

Building Our Own Number System: Base 5

The Babylonians used base 60 —an unusual but very powerful place-value system. Let' s try to build a simpler one ourselves: a **base 5 system**.

Digits

In base 5, each "place" represents a power of 5 instead of 10. That means we only need the digits

$$0, \ 1, \ 2, \ 3, \ 4.$$

Once we reach 4, the next number is written as 10_5, which means "one group of 5 and zero ones."

Place values

Place value	5^4	5^3	5^2	5^1	5^0
Value in base 10	625	125	25	5	1

Just like in base 10, the rightmost digit counts "ones," the next counts "fives," then "twenty-fives," and so on.

Reading a base 5 number

To read a base 5 number, multiply each digit by the power of 5 it represents and add:

$$\text{Example: } (243)_5 = 2 \times 25 + 4 \times 5 + 3 = 50 + 20 + 3 = 73_{10}.$$

Writing a base 10 number in base 5

To convert a normal (base 10) number into base 5, repeatedly divide by 5 and record the remainders.

$$73 \div 5 = 14 \text{ remainder } 314 \div 5 = 2 \text{ remainder } 42 \div 5 = 0 \text{ remainder } 2$$

Reading remainders from bottom to top gives $(243)_5$.

A few examples

Base 10	Base 5
1	1
4	4
5	10
9	14
10	20
25	100
37	122
125	1000

Why this works. Base 5 follows exactly the same principle as our familiar base 10: each column represents a power of the base. The only difference is which powers we use.

$$\text{Base 10: } 10^0, 10^1, 10^2, 10^3, \ldots \text{ Base 5: } 5^0, 5^1, 5^2, 5^3, \ldots$$

Every whole number can be written in base 5 — it just uses fewer symbols, and each "place" grows five times larger instead of ten.

Counting in Base 5

Base-10	Base-5	Base-10	Base-5	Base-10	Base-5	Base-10	Base-5	Base-10	Base-5
0	0	10	20	20	40	30	110	40	130
1	1	11	21	21	41	31	111	41	131
2	2	12	22	22	42	32	112	42	132
3	3	13	23	23	43	33	113	43	133
4	4	14	24	24	44	34	114	44	134
5	10	15	30	25	100	35	120	45	140
6	11	16	31	26	101	36	121	46	141
7	12	17	32	27	102	37	122	47	142
8	13	18	33	28	103	38	123	48	143
9	14	19	34	29	104	39	124	49	144

Arithmetic in Base 5

We can add and multiply numbers directly in base 5, using the same logic as in base 10 — we just regroup (carry) whenever we reach 5 instead of 10.

Addition. Let's add $(243)_5 + (132)_5$.:

$$\begin{array}{rr} & 2\,4\,3_5 \\ + & 1\,3\,2_5 \\ \hline & 4\,3\,0_5 \end{array}$$

Starting from the right:

- $3 + 2 = 5 \Rightarrow 0$ with a carry of 1.
- $4 + 3 + 1(\text{carry}) = 8 \Rightarrow 3$ with a carry of 1 (since $8 = 5 + 3$).
- $2 + 1 + 1(\text{carry}) = 4$.

So the result is $(430)_5$. Check by converting:

$$(243)_5 = 73_{10}, \quad (132)_5 = 42_{10}, \quad (430)_5 = 115_{10}.$$

It works!

Multiplication. Multiply $(13)_5 \times (4)_5$.

Think: $3_5 \times 4_5 = 12_{10} = 22_5$ (since $2 \times 5 + 2 = 12$). Carry the 2 into the next column: $4_5 \times 1_5 + 2_5 = 6_{10} = 11_5$

$$(13)_5 \times (4)_5 = (112)_5.$$

Verification: $(13)_5 = 8_{10},\ (4)_5 = 4_{10},\ (112)_5 = 32_{10}$.

Carrying and Borrowing. In subtraction, borrow a *group of 5* instead of a group of 10.

$$(204)_5 - (13)_5 = \begin{array}{r} 2\,0\,4_5 \\ -\quad 0\,1\,3_5 \\ \hline 1\,4\,1_5 \end{array}$$

Everything works the same way — only the base changes!

Practice: Base 5 Numbers

1. Write these base 5 numbers in base 10:

$$(13)_5, \quad (204)_5, \quad (4013)_5, \quad (1002)_5.$$

2. Convert these base 10 numbers into base 5:

$$7, \quad 18, \quad 65, \quad 123.$$

3. What pattern do you notice when counting in base 5 from 1 to 20?

Challenge: Imagine an alien civilization that uses base 5 because they have *five fingers total*. How would they write 2025_{10} in their own system?

Binary Numbers (Base 2)

Why Binary?

Our number system is based on ten digits (0-9) because we have ten fingers. But computers don't have fingers — they have tiny electronic switches that can only be **on** or **off**.

These two states can be represented by just two digits:

$$\text{OFF} = 0, \qquad \text{ON} = 1.$$

Because of this, computers use the **binary system** (base 2), which has only two symbols: 0 and 1.

Two-State Logic

In electronics:

- High voltage → 1
- Low voltage → 0

Every circuit, image, and sound on your device is ultimately made of long chains of these two values!

Place Values in Binary

Just like base 10 uses powers of 10, binary uses powers of 2.

Place	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0
Value (base 10)	128	64	32	16	8	4	2	1

Reading a Binary Number

Each position tells you how many of that power of 2 you have.

$$(1011)_2 = 1 \times 8 + 0 \times 4 + 1 \times 2 + 1 \times 1 = 11_{10}.$$

Writing a Number in Binary

To convert from base 10 to binary, repeatedly divide by 2 and record the remainders.

$$25 \div 2 = 12 \text{ R1}, \quad 12 \div 2 = 6 \text{ R0}, \quad 6 \div 2 = 3 \text{ R0}, \quad 3 \div 2 = 1 \text{ R1}, \quad 1 \div 2 = 0 \text{ R1}.$$

Reading remainders from bottom to top gives $(11001)_2$.

Quick Check

$$(11001)_2 = 1 \times 16 + 1 \times 8 + 0 \times 4 + 0 \times 2 + 1 \times 1 = 25_{10}.$$

Always test your conversions both ways!

Counting in Binary

10	2	10	2	10	2	10	2
0	0	8	1000	16	10000	24	11000
1	1	9	1001	17	10001	25	11001
2	10	10	1010	18	10010	26	11010
3	11	11	1011	19	10011	27	11011
4	100	12	1100	20	10100	28	11100
5	101	13	1101	21	10101	29	11101
6	110	14	1110	22	10110	30	11110
7	111	15	1111	23	10111	31	11111

Side Note: Bit, Byte, and Beyond

- One binary digit = a **bit**.
- 8 bits = 1 **byte**.
- 1 byte can represent $2^8 = 256$ different values.

That's enough for all English letters, numbers, and symbols! Images, music, and videos use thousands or millions of bytes.

Binary in Everyday Life

- **Text:** Each letter or symbol is stored as a number code (for example, A = 65 in ASCII, or 01000001 in binary).
- **Color:** Each pixel on a screen is described by three numbers — the intensities of red, green, and blue light. Typically, each intensity uses 1 byte (8 bits), so a single pixel uses $3 \times 8 = 24$ bits of binary data. For example:

  $$\text{Red} = 11111111, \quad \text{Green} = 00000000, \quad \text{Blue} = 00000000$$

  makes a pure red pixel.
- **Sound:** A sound wave is recorded by measuring its height (amplitude) many times per second.

Each measurement is converted into a binary number — usually 16 bits or more per sample — so digital audio is just a very long sequence of binary numbers representing how the air vibrates over time.

All digital information — text, pictures, and sound — is ultimately stored as long strings of 0s and 1s.

Arithmetic in Binary

Binary arithmetic is especially important — it's what computers actually do! Every operation (addition, subtraction, multiplication) follows the same regrouping rules as base 10, but with only two digits: 0 and 1.

Addition. Add $(1011)_2 + (1101)_2$.

$$\begin{array}{rr} & 1\,0\,1\,1_2 \\ + & 1\,1\,0\,1_2 \\ \hline & 1\,1\,0\,0\,0_2 \end{array}$$

Work from right to left:

- $1 + 1 = 10_2 \to$ write 0, carry 1.
- Next: $1 + 0 + 1(\text{carry}) = 10_2 \to$ write 0, carry 1.
- Next: $0 + 1 + 1(\text{carry}) = 10_2 \to$ write 0, carry 1.
- Next: $1 + 1 + 1(\text{carry}) = 11_2 \to$ write 1, carry 1 to a new place.

Result:

$$(1011)_2 + (1101)_2 = (11000)_2.$$

Check: $11_{10} + 13_{10} = 24_{10}$, and indeed $24_{10} = (11000)_2$.

Multiplication. Multiply $(101)_2 \times (11)_2$.

$$\begin{array}{rr} & 1\,0\,1_2 \\ \times & 1\,1_2 \\ \hline & 1\,0\,1_2 \\ + & 1\,0\,1\,0_2 \\ \hline & 1\,1\,1\,1_2 \end{array}$$

Explanation: $(101)_2 = 5_{10},\ (11)_2 = 3_{10},\ (1111)_2 = 15_{10}$. Perfect!

Borrowing in Subtraction. Subtraction in base 2 works the same way: if you need to subtract 1 from 0, borrow from the next place (which is worth 2).

$$(1000)_2 - (1)_2 = (111)_2,$$

because $8 - 1 = 7$.

Key Patterns.

- $1 + 0 = 0 + 1 = 1_2 \quad$ (write 1)
- $1 + 1 = 10_2 \quad$ (write 0, carry 1)
- $1 + 1 + 1 = 11_2 \quad$ (write 1, carry 1)
- $10_2 + 10_2 = 100_2 \quad$ (each power of two doubles neatly)

That's all the rules a computer needs to perform any arithmetic operation!

Practice: Binary Numbers

1. Convert to base 10:

$$(1101)_2, \quad (10010)_2, \quad (111111)_2.$$

2. Convert to base 2:

$$9, \quad 14, \quad 25, \quad 42.$$

3. What happens when you count from 0 to 7 in binary? Try it on your fingers: each finger can represent one binary digit!

Challenge

1. What is the binary representation of 1000_{10}?
2. Which binary number is twice as large as $(1011)_2$?
3. What pattern do you notice between binary numbers and powers of 2?

Homework

1. Write the following numbers using Egyptian symbols:

$$37, \quad 205, \quad 1{,}204, \quad 6{,}315.$$

(Hint: use the correct number of strokes, heel bones, coils, and lotus flowers.)

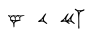
2. Write the following Egyptian numerals as regular numbers:

3. Write the following base-10 numbers in the Babylonian base-60 system (showing each "digit" as a number between 0 and 59):

$$73, \quad 125, \quad 3{,}600, \quad 3{,}726.$$

(Hint: divide by 60 repeatedly.)

4. Express these Babylonian numbers in our base-10 system:

(Example:  $= (4, 10, 21)_{60} = 4 \times 60^2 + 10 \times 60 + 21 = 15{,}021$.)

5. Write each in normal (Arabic) numbers:

XLII, CDXVI, MMXVIII, CMXC.

6. Write each in Roman numerals:

$$14, \quad 70, \quad 944, \quad 2025.$$

7. Convert numbers between numeral systems.

   (a) Convert to base 10:

   $$10101_2, \quad 2407_8, \quad 7B1A_{12}.$$

   (b) Convert 57179_{10} to base 5 and to base 9.

8. Write the numbers **245** and **324** in the **base-6** system (digits allowed: $0, 1, 2, 3, 4, 5$).

9. Convert the following **base-6** numbers to **decimal**:

   (a) 234_6

   (b) 403_6

10. Convert the following **decimal** numbers to **binary**:

$$9, \ 12, \ 24, \ 38, \ 45$$

11. Convert the following **binary** numbers to **decimal**:

$$101, \ 1001, \ 10110, \ 11011, \ 10101$$

12. Compute column-style ("by hand") in the indicated bases.

   (a) In binary:

   $$101011_2 + 111110_2, \qquad 1011_2 \times 1011_2.$$

   (b) In ternary:

   $$122121_3 + 212012_3, \qquad 1122_3 \times 120_3.$$

(c) In base 5:

$$30423_5 + 21123_5, \qquad 3214_5 \times 142_5.$$

13. Perform the following **binary operations** (without converting to decimal):

    (a) $110101_2 + 111011_2$

    (b) $10101_2 \times 1011_2$

    (c) $(10101_2 + 1101_2) \times 10110_2$

14. Perform the following in **base 4**:

    (a) $333_4 \times 2_4$

    (b) $1111_4 - 222_4$

    (c) $3231_4 - 1321_4$

15. 100 children signed up for a math club: 24 boys and 21 girls. In which base system does the club operate?

16. You have 15 water samples, exactly one contaminated. A test detects whether a *mixture* contains the chemical. Can you find the bad sample using fewer than 15 tests?

17. Place 127 one-dollar bills into seven envelopes so any amount from 1 to 127 dollars can be paid *without opening* the envelopes.

18. Robert thinks of a whole number between 1 and 1000. Julia may ask only yes/no questions (and Robert is truthful). Can she find the number in 10 questions?

19. On the board remains a half-erased sum:

```
      2 3   5
  +   1   6 4 2
  -------------
      4 2 4 2 3
```

    Determine the base of this numeral system and reconstruct the missing digits.

*20. **Divisibility rules in other bases.**

    (a) Formulate and prove a divisibility rule for 2 in the ternary system.

    (b) Formulate and prove a rule for 7 in the octal system.