# MATH 5: HANDOUT 6

## More Word Problems. Ratios

## Solving Word Problems

Today we continue practicing **word problems** and how to translate them into **equations**.

### General Strategy

1. Read the problem carefully and identify what is being asked.
2. Introduce a variable ($x$, $y$, $\dots$) for the unknown quantity.
3. Translate the words into an equation involving that variable.
4. Solve the equation. If fractions are present, clear denominators by multiplying both sides.
5. Check the solution in the original problem context.

### Examples

**Example 1 (Work problem).** Alice can paint a room in 6 hours, Bob in 4 hours. How long if they work together?

*Solution.* Alice paints $\frac{1}{6}$ of the room per hour, Bob paints $\frac{1}{4}$. Together they paint $\frac{1}{6}+\frac{1}{4}=\frac{5}{12}$ per hour. Thus the job takes $\frac{12}{5}=2.4$ hours.

**Example 2 (Distance problem).** A car travels $150$ miles at a constant speed.

1. If the speed is $50$ mph, how long does the trip take?
2. If the trip takes $2.5$ hours, what is the car's speed?

*Solution.* We use distance $=$ speed $\times$ time, so

$$t=\frac{d}{v} \quad \text{and} \quad v=\frac{d}{t}.$$

1. $t=\dfrac{150}{50}=3$ hours.
2. $v=\dfrac{150}{2.5}=60$ mph. (Since $150\div 2.5=150\cdot\frac{2}{5}=60$.)

**Example 3 (Distance problem).** Two towns are 180 miles apart. A car leaves Town A at 8:00 a.m. traveling at 60 mph. At 9:00 a.m., a bus leaves Town B heading toward Town A at 45 mph. At what time will they meet?

*Solution.* Let $t$ be the number of hours the car has traveled when they meet. In that time, the car covers $60t$ miles. The bus travels $t-1$ hours, so it covers $45(t-1)$ miles.

Together, they cover the full distance:

$$60t+45(t-1)=180.$$

Simplify:

$$60t+45t-45=180 \quad\Rightarrow\quad 105t=225 \quad\Rightarrow\quad t=\tfrac{225}{105}=\tfrac{15}{7}\approx 2.14 \text{ hours.}$$

The car started at 8:00 a.m., so they meet a little after 10:08 a.m.

**Example 4 (Age problem).** A father is 40 years older than his son. In 10 years, the father's age will be three times the son's age. How old are they now?

*Solution.* Let $s$ be the son's current age. Then the father's current age is $s + 40$.
In 10 years:

$$s + 10 \quad \text{and} \quad s + 40 + 10 = s + 50.$$

Equation:

$$s + 50 = 3(s + 10).$$

Solve:

$$s + 50 = 3s + 30 \quad \Rightarrow \quad 20 = 2s \quad \Rightarrow \quad s = 10.$$

So the son is 10, and the father is $10 + 40 = 50$.

**Key Takeaways**

- Multiply through by the least common denominator (LCD).
- Always distribute carefully.
- Check that no solution makes the denominator zero.

# Ratios

## Why do we need ratios?

Peter has $10 more than Robert. Is that a big difference?

It depends on how much money they each have.

| Peter | Robert | Ratio (Peter : Robert) |
|---|---|---|
| $12 | $2 | $\frac{12}{2} = 6 : 1$ |
| $112 | $102 | $\frac{112}{102} \approx 1.10 : 1$ |
| $1,112 | $1,102 | $\frac{1112}{1102} \approx 1.009 : 1$ |

In every case, the absolute difference is exactly $10. But notice how *different* these situations feel:

- When Robert has only $2, Peter's amount is **six times larger** — a huge difference.
- When Robert has $102, Peter has about **10% more** — noticeable, but not dramatic.
- When Robert has $1,102, Peter has only about **1% more** — almost the same.

The reason is that our sense of "big" or "small" difference often depends not on the subtraction ($10), but on the *ratio* between the two amounts. That ratio tells us how large one quantity is *relative to* another.

A **ratio** compares two quantities by division:

$$\text{ratio of } a \text{ to } b = \frac{a}{b} \quad \text{or} \quad a : b \quad \text{or} \quad \text{“a to b”}.$$

Peter has $12, Robert has $2. The ratio of Peter's money to Robert's money is

$$\frac{12}{2} = 6 \quad \text{or} \quad 6 : 1.$$

This means Peter has 6 times as much money as Robert.

## Interpreting Ratios and Equivalent Ratios

Let's look at a simple and tasty example — making lemonade!

The recipe says that the ratio of **water : lemon juice** is $4 : 1$.

That means that for every 1 part of lemon juice, we need 4 parts of water:

$$\frac{\text{water}}{\text{juice}} = \frac{4}{1}.$$

The total number of parts is $4 + 1 = 5$. So:

$$\text{lemon juice} = \tfrac{1}{5} \text{ of the total}, \qquad \text{water} = \tfrac{4}{5} \text{ of the total}.$$

| **Total Volume** | **Juice Needed** | **Water Needed** |
|---|---|---|
| 1 L | 0.2 L | 0.8 L |
| 1.5 L | 0.3 L | 1.2 L |
| 2 L | 0.4 L | 1.6 L |

This shows how easy it is to scale the recipe up or down when the ratio is known.

Ratios help us describe the *relative amounts* of ingredients or quantities. They do not depend on the actual units — only on how much one quantity is compared to another.

Now, suppose we want to make the lemonade sweet. We add sugar!

If we add sugar, the recipe might say:

$$\text{water} : \text{lemon juice} : \text{sugar} = 4 : 1 : 0.5.$$

That means:

- For every 1 part of lemon juice, use 4 parts of water.
- For every 1 parts of lemon juice, use 0.5 part of sugar.

The total number of "parts" is $4 + 1 + 0.5 = 5.5$.

However, the ratio $4 : 1 : 0.5$ has a fraction, which is not very convenient to work with. We can make all numbers whole by multiplying each term by 2:

$$4 : 1 : 0.5 = 8 : 2 : 1.$$

Now the ratio tells us that for every 1 part of sugar, we need 2 parts of lemon juice and 8 parts of water — a nice, simple whole-number ratio.

In fact, ratios can be scaled up or down just like fractions. Such ration are called **equivalent**.

$$4 : 1 = 8 : 2 = 12 : 3.$$

**Example** Sweet lemonade uses water : juice : sugar = 8 : 2 : 1. However, the ratio of water : juice is the same as in unsweetened lemonade: $8 : 2 = 4 : 1$

**Equivalent ratios** represent the same relationship. Multiplying or dividing all terms of a ratio by the same number gives an equivalent ratio.

## Ratios in word problems

Irene has 1686 balloons — red, blue, and green. The ratio of red : blue is 2 : 3. She sells $\frac{3}{4}$ of the blue balloons, $\frac{1}{2}$ of the green balloons, and none of the red ones. After selling, she has 922 balloons left. How many blue balloons did she have at first?

**Solution (sketch).**

1. Let red $= 2x$, blue $= 3x$, green $= g$.
2. Total before sale: $2x + 3x + g = 1686$.
3. After sale: $2x + \frac{1}{4}(3x) + \frac{1}{2}g = 922$.
4. Solve:
$$5x + g = 1686, \quad 2x + \tfrac{3}{4}x + \tfrac{1}{2}g = 922.$$
   Simplify second: $\frac{11}{4}x + \frac{1}{2}g = 922$.
5. Multiply by 4: $11x + 2g = 3688$. Subtract 2×(first): $11x + 2g - 2(5x + g) = 3688 - 3372$, so $x = 316$.
6. Hence blue $= 3x = 948$, red $= 632$, green $= 106$.

**Key Takeaways**

- Ratios compare quantities by division.
- Equivalent ratios show the same relationship.

## Classwork: Ratios and Proportions

1. Five math classes a day—is that a lot? Five math classes a year—is that a lot?
2. If we increase the weight of an ant by $1$ g, will that be a significant change (given that an average ant weighs less than $10$ mg)? What if we increase the weight of an elephant by $1$ g? Come up with your own examples where increasing a quantity by the same amount produces very different effects.
3. From a $20$-liter can of milk, $6$ liters were poured out. What is the ratio of the amount poured out to the amount remaining?
4. The ratio of cashews to walnuts in a nut mixture is $2 : 3$. The total weight of the mixture is $150$ g. How many grams of cashews and walnuts are in the mixture?
5. In a dried fruit mix, there are $7$ parts of dried apples, $4$ parts of dried pears, and $5$ parts of dried apricots. If the total weight of the mix is $1600$ g, find the weight of each type of fruit.

## Solutions: Ratios and Proportions

1. Five math classes a day—is that a lot? Five math classes a year—is that a lot?

   **Solution:** ”Is it a lot?” depends on context and scale.

   Five math classes in one day is a lot, because in one school day you usually have only a few class periods total. That would be most of your day.

   But five math classes in an entire year is very little, because a school year has on the order of 150–180 school days. Most students have math almost every day, so they would have well over 100 math classes per year.

   So: the same number (5) can be ”a lot” or ”not a lot” depending on what it is being compared to.

2. If we increase the weight of an ant by $1$ g, will that be a significant change (given that an average ant weighs less than $10$ mg)? What if we increase the weight of an elephant by $1$ g? Come up with your own examples where increasing a quantity by the same amount produces very different effects.

   **Solution:** An average ant weighs less than $10$ mg $= 0.01$ g. Increasing its weight by $1$ g would make it over $100$ times heavier. That is an enormous change.

   An elephant weighs a few *tons* (thousands of kilograms). Adding $1$ g to something that already weighs, say, $4000$ kg is basically no change at all — it’s undetectable.

   So the same increase ($+1$ g) is huge in one case and irrelevant in the other, because of scale.

   **Example:** Adding $10 to someone’s money:

   - If a person has $12, getting $10 is a huge deal (their money almost doubles).
   - If a billionaire has $5,000,000,000, getting $10 makes no practical difference.

   Another example: raising the temperature of a room by $5°$C is a big comfort change; raising the temperature of the Sun by $5°$C is basically nothing compared to millions of degrees.

3. From a $20$-liter can of milk, $6$ liters were poured out. What is the ratio of the amount poured out to the amount remaining?

   **Solution:** Amount poured out: $6$ L.

   Amount remaining: $20 - 6 = 14$ L.

   So the ratio (poured out) : (left) is

   $$6 : 14.$$

   We usually simplify ratios the same way we simplify fractions. Divide both numbers by $2$:

   $$6 : 14 = 3 : 7.$$

   **Answer:** $3 : 7$.

4. The ratio of cashews to walnuts in a nut mixture is $2 : 3$. The total weight of the mixture is $150$ g. How many grams of cashews and walnuts are in the mixture?

   **Solution:** The ratio $2 : 3$ means: for every $2$ “parts” of cashews, there are $3$ “parts” of walnuts.

   Total parts $= 2 + 3 = 5$ parts.

   Each part, in grams, is

   $$\frac{150 \text{ g}}{5} = 30 \text{ g}.$$

   So:

   $$\text{cashews} = 2 \times 30 \text{ g} = 60 \text{ g}, \qquad \text{walnuts} = 3 \times 30 \text{ g} = 90 \text{ g}.$$

   **Answer:** $60$ g cashews and $90$ g walnuts.

5. In a dried fruit mix, there are $7$ parts of dried apples, $4$ parts of dried pears, and $5$ parts of dried apricots. If the total weight of the mix is $1600$ g, find the weight of each type of fruit.

   **Solution:** The ratio is $7:4:5$.

   Total number of parts is

   $$7 + 4 + 5 = 16 \text{ parts.}$$

   Each part weighs

   $$\frac{1600 \text{ g}}{16} = 100 \text{ g.}$$

   Therefore:

   $$\text{apples} = 7 \times 100 \text{ g} = 700 \text{ g}, \qquad \text{pears} = 4 \times 100 \text{ g} = 400 \text{ g}, \qquad \text{apricots} = 5 \times 100 \text{ g} = 500 \text{ g.}$$

   Quick check: $700 + 400 + 500 = 1600$, matches.

   **Answer:** apples $700$ g, pears $400$ g, apricots $500$ g.

## Homework

1. **Simplify the following expressions:**

   (a) $\frac{2}{3}x + \frac{4}{3}(1+x)$ (b) $2.5x - 1.5(4-x)$ (c) $2\left(x - \frac{2}{3}\right) - \left(x + \frac{1}{2}\right)$

2. **Solve the following equations:**

   (a) $\frac{3}{4}(x+8) = 10$ (b) $\frac{1}{2}(x+1) = x - 3$ (c) $\frac{1}{2}x + \frac{1}{3}x = x - \frac{1}{12}$

3. **Solve the following equations:**

   (a) $\dfrac{x}{8} = \dfrac{3}{12}$ (b) $\dfrac{5}{x} = \dfrac{15}{18}$ (c) $\dfrac{x-2}{6} = \dfrac{4}{9}$ (d) $\dfrac{7}{x+3} = \dfrac{14}{10}$

4. Pirate captain John can drink a barrel of rum in 14 days. If he drinks together with pirate Bill, they finish the barrel in 10 days. How long would it take Bill to drink the barrel of rum alone?

5. A truck covers the distance between two cities in 10 hours. A fast car, which goes 10 mph faster than the truck, covers the same distance in 8 hours. What is the distance? *Hint: If the truck speed is $x$ mph, then the distance is $10x$ miles.*

6. If you take half my age and add 7, you get my age 13 years ago. How old am I?

7. One can measure temperature using either the Fahrenheit scale (used in the US and Britain) or the Celsius scale (used in most other countries). The relation is

   $$C = \tfrac{5}{9}(F - 32),$$

   where $C$ is Celsius and $F$ is Fahrenheit.

   (a) Find a temperature that has the same value on both scales ($F = C$).

   (b) Find a temperature where the Fahrenheit reading is twice the Celsius reading ($F = 2C$).

8. The ratio of roses to hibiscuses in a garden is $9 : 11$. If there are $99$ rose bushes, how many hibiscus bushes are there? What is the total number of bushes?

9. Mary prepared a homemade dried fruit and nut mix using $6$ parts of raisins, $5$ parts of dried cranberries, and $3$ parts of walnuts. The cranberries and walnuts together weighed $2$ kg $400$ g. Find the total weight of the mix.

10. Sea water contains $5\%$ salt by weight. How many kilograms of fresh water must be added to $40$ kg of sea water to obtain a solution with $2\%$ salt?

11. Bronze is an alloy of tin and copper. The ratio of tin to copper in bronze is $3 : 17$. How much tin and how much copper are in an $80$ kg piece of bronze?

12. Fresh watermelon weighs $10$ kg and contains $99\%$ water. After losing some water in the store, it now contains only $98\%$ water. What is its new weight?

13. Solve the following puzzle (different letters stand for different digits):

$$\begin{array}{r} \text{THIS} \\ +\ \ \text{IS} \\ \hline \text{EASY} \end{array}$$