MATH 5: HANDOUT 5

ALGEBRAIC EXPRESSIONS AND FRACTIONS

Division and Fractions

So far we have studied the rules of addition, subtraction, and multiplication. Now we complete the picture with **division and fractions**.

The key idea is that division can always be written as a fraction:

$$a \div b = \frac{a}{b}.$$

Rules for Fractions

For any nonzero numbers b, d:

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}, \qquad \frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}.$$

When the numerator has addition or subtraction:

$$\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}, \qquad \frac{a-b}{c} = \frac{a}{c} - \frac{b}{c}.$$

Examples

1. Simplify $a \div (b \div c)$:
$$a \div \frac{b}{c} = a \times \frac{c}{b} = \frac{ac}{b}.$$
2. Solve $\frac{5}{7}x = 15$. Multiply both sides by $\frac{7}{5}$:
$$x = 15 \times \frac{7}{5} = \frac{105}{5} = 21.$$
3. Compute $\frac{2}{3} \div \frac{5}{6}$:
$$\frac{2}{3} \times \frac{6}{5} = \frac{12}{15} = \frac{4}{5}.$$

Common Mistakes to Avoid

- Remember that $\frac{a}{b+c} \neq \frac{a}{b} + \frac{a}{c}$.
- Division by zero is undefined.
- Always reduce fractions when possible.

Classwork

1. There are fewer than 50 students in the class. On a test, one-seventh of the students received As, one-third received Bs, and one-half received Cs. The rest received failing grades. How many students received failing grades?

   Solution Let the total number of students be N. Then

   $$\frac{1}{7}N + \frac{1}{3}N + \frac{1}{2}N + x = N,$$

   where x is the number of failing grades.

   Simplify:

   $$x = N\left(1 - \frac{1}{7} - \frac{1}{3} - \frac{1}{2}\right).$$

   Find a common denominator of 42:

   $$\frac{1}{7} + \frac{1}{3} + \frac{1}{2} = \frac{6 + 14 + 21}{42} = \frac{41}{42}.$$

   Therefore,

   $$x = N\left(1 - \frac{41}{42}\right) = N \cdot \frac{1}{42}.$$

   For x to be a whole number, N must be a multiple of 42. Since there are fewer than 50 students, we have $N = 42$.

   Then

   $$x = \frac{42}{42} = 1.$$

   Answer There were 42 students in the class, and **1 student received a failing grade**.

2. Nadya baked tarts with raspberry, blueberry, and strawberry fillings. Raspberry tarts made up half of all tarts; there were 14 fewer blueberry tarts than raspberry tarts; and the number of strawberry tarts was half of the combined number of raspberry and blueberry tarts. How many tarts of each type did she bake?

   Solution Let T be the total number of tarts, and let R, B, S denote raspberry, blueberry, and strawberry tarts, respectively. We are told

   $$R = \tfrac{1}{2}T, \qquad B = R - 14, \qquad S = \tfrac{1}{2}(R + B).$$

   Then $S = \frac{1}{2}(R + R - 14) = R - 7$, so

   $$T = R + B + S = R + (R - 14) + (R - 7) = 3R - 21.$$

   Since $T = 2R$, we get $2R = 3R - 21$, hence $R = 21$. Therefore

   $$B = 21 - 14 = 7, \qquad S = 21 - 7 = 14, \qquad T = 42.$$

   Answer Raspberry 21, Blueberry 7, Strawberry 14.

3. If Anne walks to school and takes the bus back, the total travel time is 1.5 hours. If she takes the bus both ways, the total time is 30 minutes. How long will her total travel time be if she walks both ways?

Solution The time for walking both ways and taking the bus both ways is the same as the time for two trips when you walk one way and take the bus the other way.

$$(\text{walk} + \text{bus}) + (\text{walk} + \text{bus}) = 2 \cdot 1.5 = 3.$$

At the same time, we can rewrite this as:

$$(\text{walk} + \text{walk}) + (\text{bus} + \text{bus}) = 3.$$

Since $\text{bus} + \text{bus} = .5$, it follows that $\text{walk} + \text{walk} = 2.5$.

Answer 2.5 hours.

4. Using five 3s, arithmetic operations, and parentheses, make five expressions that are equal numbers from 1 to 5.

Solution Note, that some numbers allow several different expressions!

$$\begin{aligned}
1 &= (3 \times 3 + 3) \div 3 - 3 \\
2 &= (3 + 3 + 3 - 3) \div 3 \\
3 &= (3 \times 3 + 3 - 3) \div 3 = 3 \times 3 \times 3 \div 3 \div 3 \\
4 &= (3 + 3 + 3 + 3) \div 3 \\
5 &= (3 \times 3 + 3 + 3) \div 3
\end{aligned}$$

Homework

1. **Compute:**
   (a) $5 \div 7\frac{1}{2}$
   (b) $(2 - 3.5) \times \frac{2}{3}$
   (c) $3\frac{2}{11} \div \frac{5}{7}$

2. **Simplify each expression.**
   (a) $x + 4(1 - x)$
   (b) $2 + 5x - 4(3 - x)$
   (c) $5(x - 1) - 3(2x + 1)$
   (d) $x - \frac{4}{5}(1 - x)$
   (e) $\left(\frac{2}{3}x + 1\right) \div \frac{6}{7}$
   (f) $2 \div (3 \div x - 1)$

3. **Solve the equations.**
   (a) $\frac{3}{4}x = 12$
   (b) $\frac{1}{2}x + \frac{1}{7}x = 18$
   (c) $\frac{2}{3}(x + 7) = 12$
   (d) $2.1x + \frac{1}{5}x + 1 = 9$

4. John bought a large bag of red, green, and blue candies, 74 candies in all. The number of red candies is one less than the number of green candies, and there are as many blue candies as red and green together. How many candies of each color are there?

5. A boy had a bag of apples. He gave $\frac{1}{2}$ of them to his parents, $\frac{1}{5}$ to his brother, $\frac{1}{4}$ to his sister, and ate the last apple himself. How many apples did he originally have?

6. Jane is 5 years old and her father is 42. In how many years will he be twice as old as she? Three times as old? (Hint: denote the number of years by x and write equations for x.)

7. A hot water tap fills the bath in 5 minutes. The cold water tap fills the bath in 3 minutes. With both taps open, how long will it take to fill the bath?

8. A watermelon is 99% water (by weight). Watermelon jelly is 98% water. How much water must evaporate from 1 kg of watermelon to turn it into jelly?

9. Using five 3s, arithmetic operations, and parentheses, make five expressions that equal to numbers from 6 to 10.

10. We'll call a number "lucky" if, between its digits, we can insert arithmetic operation signs ($+$, $-$, $\times$, $\div$) and parentheses so that the resulting expression equals 100. Is number 123456 lucky?

*11. Forty children were holding hands in a circle. Among them, 22 were holding a boy's hand and 30 were holding a girl's hand. How many girls were there in the circle?

*12. The numbers $1, 2, \ldots, 10$ are written in a row. Is it possible to put signs $+$ and $-$ between them so that the result is zero?