MATH 5: HANDOUT 3

ALGEBRAIC EXPRESSIONS

Algebraic Expressions and Basic Laws

Variables and Algebraic Expressions

In mathematics and the sciences we often use letters instead of numbers. This allows us to describe general relationships that hold for all numbers, or to represent unknown values. Such letters are called **variables**.

Expressions that involve both numbers and variables are called **algebraic expressions**.

Examples:

$$3a, \quad 7b+8, \quad 357+10x, \quad \frac{65z-459}{4}$$

When writing algebraic expressions, we omit the multiplication sign between a number and a variable. Instead of $7 \times b$ we write $7b$. In products, the number always comes first: we write $10k$ instead of $k \times 10$.

Fundamental Laws of Arithmetic

The following laws allow us to rearrange and simplify expressions:

$$\begin{aligned}
a+b &= b+a && \text{(Commutative law of addition)}\\
a+(b+c) &= (a+b)+c && \text{(Associative law of addition)}\\
ab &= ba && \text{(Commutative law of multiplication)}\\
a(bc) &= (ab)c && \text{(Associative law of multiplication)}\\
a(b+c) &= ab+ac && \text{(Distributive law)}
\end{aligned}$$

Examples:

- Commutative: $4+7=7+4$ or $2x \cdot 3y = 3y \cdot 2x$
- Associative: $(1+2)+3=1+(2+3)$
- Distributive: $5(x+2)=5x+10$

These laws can be used for simplifying calculations and rewriting expressions in a simpler form.

Using Commutativity, Associativity, and Distributivity

Proof 1. $2(x+3)+4(x+3)=6(x+3)$

$$\begin{aligned}
2(x+3)+4(x+3) &= (2x+6)+(4x+12) && \text{Distributive}\\
&= \big((2x+6)+4x\big)+12 && \text{Associative (regroup the four-term sum)}\\
&= \big(2x+(6+4x)\big)+12 && \text{Associative (move parentheses inward)}\\
&= \big(2x+(4x+6)\big)+12 && \text{Commutative (swap 6 and } 4x\text{)}\\
&= \big((2x+4x)+6\big)+12 && \text{Associative (regroup like terms)}\\
&= (2x+4x)+(6+12) && \text{Associative (separate into two sums)}\\
&= \big((2+4)x\big)+18 && \text{Distributive (factor } x \text{ from } 2x+4x\text{)}\\
&= 6x+18 && \text{Arithmetic}\\
&= 6(x+3) && \text{Distributive (factor 6).}
\end{aligned}$$

Proof 2. $3(x+4)+2x = 5x+12$

$$\begin{aligned}
3(x+4)+2x &= (3x+12)+2x && \text{Distributive} \\
&= \big((3x+12)+2x\big) && \text{(Insert parentheses to show grouping)} \\
&= \big(3x+(12+2x)\big) && \text{Associative (shift parentheses)} \\
&= \big(3x+(2x+12)\big) && \text{Commutative (swap 12 and } 2x\text{)} \\
&= \big((3x+2x)+12\big) && \text{Associative (regroup like terms)} \\
&= \big((3+2)x\big)+12 && \text{Distributive (factor } x \text{ from } 3x+2x\text{)} \\
&= 5x+12 && \text{Arithmetic.}
\end{aligned}$$

Collecting Like Terms

Expressions can often be simplified by combining **like terms —** terms that involve the same variable raised to the same power. When we do this, we are *implicitly using the laws of arithmetic*.
Example:

$$\begin{aligned}
2x+3+5(x+1) &= 2x+3+(5x+5) && \text{(Distributive law)} \\
&= 2x+5x+3+5 && \text{(Commutative law, reorder terms)} \\
&= (2x+5x)+(3+5) && \text{(Associative law, regroup)} \\
&= (2+5)x+8 && \text{(Distributive law, factor } x\text{)} \\
&= 7x+8.
\end{aligned}$$

We could combine $2x$ and $5x$ because both are multiples of x. But in an expression like $2x+7y$, we cannot combine the terms because the variables are different.

Solving Simple Equations

To solve an equation means to find the value of the variable that makes the equation true. We simplify step by step, using the following principles:

- We may add or subtract the same number from both sides.
  Example: $3x+5=20 \Rightarrow 3x=15$ (obtained by subtracting 5 from both sides of the original equation)
- We may multiply or divide both sides by the same nonzero number.
  Example: $3x=15 \Rightarrow x=5$ (obtained by dividing both sides by 3)

These rules allow us to transform equations into simpler and simpler forms until the value of the variable is clear.

Example 1

$$\begin{aligned}
&2x+7=15 \\
&2x=8 \\
&x=4
\end{aligned}$$

Thus, the solution is $x=4$.

Example 2

$$\begin{aligned}
&2(x-2)+3(2-x)=17 \\
&2x-4+6-3x=17 \\
&-x+2=17 \\
&x=-15
\end{aligned}$$

Thus, the solution is $x=-15$.

Example 3

$$
\begin{aligned}
&3 - x + 2(x - 1) - x = 1 \\
&3 - x + 2x - 2 - x = 1 \\
&1 = 1
\end{aligned}
$$

This is an identity, true for all real numbers x. Any number x is a solution to this equation.

Example 4

$$
\begin{aligned}
&(3 + x) - 5(x - 8) = 3(5 - x) \\
&3 + x - 5x + 40 = 15 - 3x \\
&43 - 4x = 15 - 3x \\
&43 = 15 + x \\
&x = 28
\end{aligned}
$$

Thus, the solution is $x = 28$.

Number of Solutions to Linear Equations

A linear equation in one variable can have different kinds of solutions.

- **One solution:** For example, $2x + 5 = 11$ has the single solution $x = 3$.
- **No solution:** For example, $2x + 3 = 2x + 5$ has no solution, since the two sides can never be equal.
- **Infinitely many solutions:** For example, $3x + 2 = 3(x + \frac{2}{3})$ is true for all x, so every real number is a solution.

Homework

Please try to do as many of the problems below as you can, and bring completed solutions with you to next class (do not forget to put your name on it!). Some of these problems are similar to those we have discussed in class; some are new. It is OK if you can not solve some problem — but do not give up before making an effort, maybe putting the problem away and coming back to it later — which means you have to start the homework early.

Please always write solutions on a separate sheet of paper. Solutions should include explanations. I want to see more than just an answer: I also want to see how you arrived at this answer, and some justification why this is indeed the answer. So **please include sufficient explanations**, which should be clearly written so that I can read them and follow your arguments.

1. Simplify the following expressions.
   (a) $1\frac{7}{8} \times \frac{18}{5}$
   (b) $2\frac{4}{7} \div frac421$
   (c) $13/7 - 7/13$
2. Evaluate the given expressions for the indicated values of x.
   (a) $78 + 3x$ for $x = 8,\ x = 2.3,\ x = \frac{2}{3}$
   (b) $\dfrac{54}{x-7}$ for $x = 8.5,\ x = 13,\ x = 11$
3. Simplify each expression by collecting like terms.
   (a) $2x + 7 + 5x + 2 + 3x$
   (b) $3x + 9 + 5xy + 2xy + 3$
   (c) $2x + 16 + 10xy + 5x + 3$
   (d) $2a + 1 + 3(a + 2)$
4. Solve the following equations.
   (a) $x + 12 = 34$
   (b) $24 - x = 10$
   (c) $2x = 96$
   (d) $3x + 2 = 44$
   (e) $5(x + 4) = 45$
5. **Geometry puzzle.** Cut a triangle into 4 smaller triangles so that any two of them share a common segment (not just a single point).
6. **Combinatorics / counting puzzle.** Marina has a large bag of M&M candy with three colors: red, green, and brown. She knows that if you draw 100 pieces from the bag, there will always be at least one piece of each color. What is the maximum possible number of candies that could be in the bag?

*7. **Logic puzzle.** Below are some examples from a multiplication table in an unknown language. All of the products are numbers less than or equal to 20.

pe × nei = nei la nei
nei × hato = liomu la pe
hato × hato = nei la tano
pe × pe = nei
pe × tano = liomu
hato × $*$ = liomu la tano
$*$ × pe = liomu la nei

Determine what number should replace $*$.