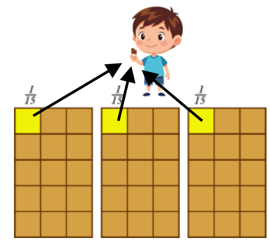
Math 4e. Class work 6.

Fractions.

A fraction (from Latin: fractus, "broken") represents a part of a whole.

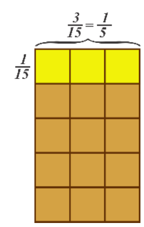
Look at the picture on the right: the whole chocolate bar is divided into 15 equal pieces:

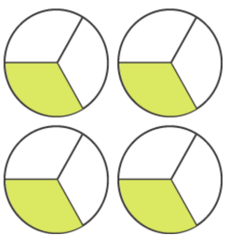


$$1\ (whole\ chocolate\ bar\ ): 15(equal\ parts) = \frac{1\ (whole\ chocolate\ bar\ )}{15(equal\ parts)}$$

$$= \frac{1}{15}(of\ the\ whole\ chocolate\ bar)$$

To divide 3 chocolate bars among 15 kids we can give each kid $\frac{1}{15}$ of each chocolate bar, altogether





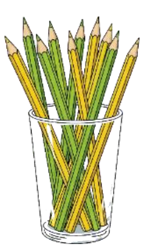
$$\frac{1}{15} + \frac{1}{15} + \frac{1}{15} = 3 \cdot \frac{1}{15} = \frac{3}{15} = \frac{1}{5} = 3: 15$$

$$3: 15 = 3 \cdot \frac{1}{15} = \frac{3}{15} = \frac{1}{5}$$

To divide 4 pizzas equally between 3 friends, we will give each friend $\frac{1}{3}$ of each pizza. Each friend will get

$4: 3 = 4 \cdot \frac{1}{3} = \frac{4}{3}$, which is exactly 1 whole pizza $(3 \cdot \frac{1}{3} = \frac{3}{3} = 1)$ and $\frac{1}{3}$.

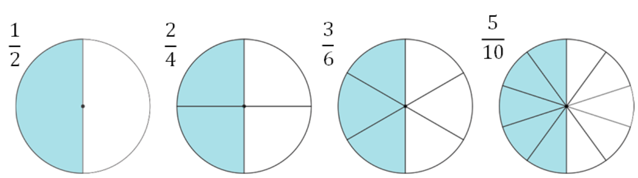
When we talk about fractions, we usually mean the part of a unit. Proper fractions are parts of a unit; improper fractions are the sums of a natural number and a proper fraction. Sometimes we want to find a part of something which is not 1, but can be considered as a single object. For example, among 30 pencils, $\frac{2}{5}$ are yellow. How many yellow pencils are there?



What does it mean to find $\frac{2}{5}$ out of 30? The whole pile of all these pencils is a single object, and we want to calculate how many pencils a little pile of $\frac{2}{5}$ of 30 contains. $\frac{2}{5}$ is 2 times $\frac{1}{5}$, and $\frac{1}{5}$ of 30 is $30: 5$. So $\frac{2}{5}$ of 30 pencils will be twice as many: $\frac{2}{5} \cdot 30 = 30: 5 \cdot 2$

Equivalent fractions.

Some fractions can look different, but represent exactly the same part of the whole.



$$\frac{1}{2}=\frac{2}{4}=\frac{3}{6}=\frac{5}{10};\qquad \frac{1}{2}=\frac{1\cdot 2}{2\cdot 2}=\frac{1\cdot 3}{2\cdot 3}=\frac{1\cdot 5}{2\cdot 5}$$

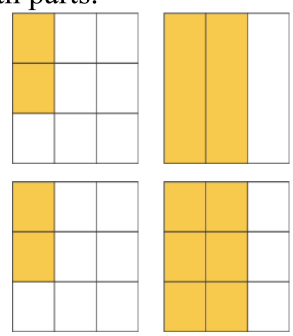
We can multiply the numerator and denominator of a fraction by the same number (not equal to 0), and the fraction will not change; it's still the same part of the whole. We are only dividing the whole into smaller parts and taking more such parts: if parts are twice smaller (denominator is multiplied by 2), we need twice more such parts to keep the fraction the same (numerator is multiplied by 2).

This property of fractions can be used to reduce fractions. If there are common factors in the numerator and denominator, both numbers can be divided by the common factors.

$$\frac{25}{35}=\frac{5\cdot 5}{7\cdot 5}=\frac{5}{7};\qquad \frac{77}{352}=\frac{7\cdot 11}{32\cdot 11}=\frac{7}{32}$$

Addition and subtraction of fractions with unlike denominators.

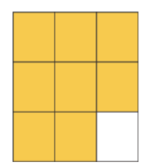
It's very easy to add fractions with the same (like) denominators. One fifteenth part oaf a chocolate bar together with another two such parts will be exactly three fifteenth parts.



$$\frac{1}{15}+\frac{2}{15}=\frac{1+2}{15}=\frac{3}{15}=\frac{1}{15}$$

Let's try to add $\frac{2}{9}$ and $\frac{2}{3}$. What should we do? Why do we need to bring both fractions to the same denominator? We can add together only similar objects: apples to apples and oranges to oranges. Are two fractions $\frac{2}{9}$ and $\frac{2}{3}$ similar objects?

$$\frac{2}{3}=\frac{1}{3}+\frac{1}{3},\qquad \frac{2}{9}=\frac{1}{9}+\frac{1}{9}$$



How we can add together

$$\frac{2}{9}+\frac{2}{3}=\frac{1}{9}+\frac{1}{9}+\frac{1}{3}+\frac{1}{3}$$

To be able to add two fractions we have to be sure that they have the same denominator. Each $\frac{1}{3}$ is exactly the same as $\frac{3}{9}$ and $\frac{2}{3}=\frac{6}{9}$

$$\frac{2}{3}=\frac{2\cdot 3}{3\cdot 3}=\frac{6}{9}$$

$$\frac{2}{9}+\frac{2}{3}=\frac{2}{9}+\frac{6}{9}=\frac{8}{9}$$

Mutually prime numbers are the numbers which do not have common factors, but 1. Like 8 and 9, are both not prime, but do not have common factors other than 1.

If we multiply both numerator and denominator by the same number, the fraction will not change. Common denominator of both fractions should be the multiple of these denominators. If both numbers are prime (or mutually prime), the least common multiple is their product. If this is not the case, least common multiple is the simplest common denominator, but not the only one, any other multiple can do this task. Numerator and denominator of each fraction should be multiplied by a corresponding number to bring both fractions to a common denominator.
For example,

$$\frac{3}{8}+\frac{5}{12}$$

Common denominator can be $8\cdot 12 = 96$, but 24 is smaller.

$$\frac{3\cdot 3}{8\cdot 3}+\frac{5\cdot 2}{12\cdot 2}=\frac{9}{24}+\frac{10}{24}=\frac{19}{24}$$

How can fractions be compared?
How can one know which fraction is grater and which is smaller? There are several ways to find out. First, fractions can be brought to a common denominator. For example, let's compare $\frac{7}{12}$ and $\frac{10}{18}$. The first fraction, $\frac{7}{9}$ is non-reducible. The second fraction can be reduced (it's a good idea to reduce fractions before doing anything):

$$\frac{10}{18}=\frac{2\cdot 5}{2\cdot 9}=\frac{5}{9}$$

Now, $\frac{7}{12}$ and $\frac{5}{9}$ can be brought to common denominator:

$$\frac{7}{12}=\frac{7\cdot 3}{12\cdot 3}=\frac{21}{36};\quad \frac{5}{9}=\frac{5\cdot 4}{9\cdot 4}=\frac{20}{36};\qquad \frac{21}{36}>\frac{20}{36}$$

The whole was divided into 36 equal parts and 21 such parts are greater than 20.
Another way to do it, is to bring them to a common numerator. Since 5 and 7 are both prime numbers, the LCM of them is their product:

$$\frac{7}{12}=\frac{7\cdot 5}{12\cdot 5}=\frac{35}{60};\quad \frac{5}{9}=\frac{5\cdot 7}{9\cdot 7}=\frac{35}{63};\qquad \frac{35}{60}>\frac{35}{63}$$

An equal number of parts are compared, but each part in the second case is smaller.
Also, both fractions can be compared with a third number, for example, $\frac{1}{2}$.

$$\frac{7}{12}=\frac{6}{12}+\frac{1}{12};\qquad \frac{10}{18}=\frac{9}{18}+\frac{1}{18}$$

Since, $\frac{7}{12}$ is greater than $\frac{1}{2}$ by $\frac{1}{12}$; and $\frac{10}{18}$ is greater than $\frac{1}{2}$ by $\frac{1}{18}$, $\frac{1}{12}>\frac{1}{18}$, so $\frac{7}{12}>\frac{10}{18}$.

Exercises.

1. Mark the following fractions on the number line (draw the number line in your notebook):

$$\frac{1}{5}, \quad \frac{3}{5}, \quad \frac{3}{3}, \quad \frac{7}{5}, \quad \frac{10}{5}$$

2. Fill the empty spaces for fractions:

$$\frac{2}{3} = \frac{}{9} = \frac{}{21} = \frac{4}{} = \frac{36}{}$$

3. Show that fractions are equal:

Example:

$$\frac{1}{11} = \frac{1 \cdot 3}{11 \cdot 3} = \frac{3}{33}$$

$a.\ \frac{1}{2} = \frac{5}{10};$ $\quad b.\ \frac{1}{5} = \frac{2}{10};$ $\quad c.\ \frac{1}{4} = \frac{5}{20};$ $\quad d.\ \frac{1}{4} = \frac{25}{100};$ $\quad e.\ \frac{1}{25} = \frac{4}{100}$

$f.\ \frac{1}{25} = \frac{3}{75};$ $\quad g.\ \frac{1}{50} = \frac{2}{100};$ $\quad h.\ \frac{1}{20} = \frac{5}{100};$ $\quad i.\ \frac{1}{22} = \frac{5}{110};$ $\quad j.\ \frac{1}{3} = \frac{6}{18}$

4. Simplify fractions:

$a.\ \frac{4}{8}; \quad \frac{9}{36}; \quad \frac{7}{28}; \quad \frac{5}{35};$ $\qquad b.\ \frac{2}{8}; \quad \frac{8}{24}; \quad \frac{12}{28}; \quad \frac{45}{100};$

$c.\ \frac{25}{35}; \quad \frac{45}{54}; \quad \frac{8}{400}; \quad \frac{32}{256};$ $\qquad d.\ \frac{12}{18}; \quad \frac{18}{20}; \quad \frac{20}{24}; \quad \frac{24}{30};$

5. Bring the fractions to the common denominator:

$a.\ \frac{3}{5}\ and\ \frac{2}{3};$ $\qquad b.\ \frac{3}{4}\ and\ \frac{5}{16};$ $\qquad c.\ \frac{1}{4}\ and\ \frac{1}{6};$

6. Compare:

$a.\ \frac{3}{5}\ and\ \frac{4}{7};$ $\qquad b.\ \frac{3}{5}\ and\ \frac{3}{8};$ $\qquad c.\ \frac{3}{6}\ and\ \frac{1}{2};$

$d.\ \frac{1}{5}\ and\ \frac{5}{1};$ $\qquad e.\ \frac{4}{12}\ and\ \frac{3}{4};$ $\qquad f.\ \frac{2}{11}\ and\ \frac{1}{12};$

7. Evaluate:

$a.\ \frac{1}{5} + \frac{1}{2};$ $\qquad b.\ \frac{2}{5} + \frac{3}{10};$ $\qquad c.\ \frac{5}{9} - \frac{1}{3};$