Math 4b. Class work 4.

Prime factorization.

In mathematics, factorization is a decomposition of a number or mathematical expression as a product of numbers or/and expressions.

A number can be represented as the product of two or more other numbers, for example:

$$40 = 4 \cdot 10 = 4 \cdot 2 \cdot 5, \quad 36 = 6 \cdot 6 = 2 \cdot 3 \cdot 6$$

A numerical expression can be written as a product:

$$7 \cdot 5 + 7 \cdot 3 = 7 \cdot (5 + 3)$$

Is it possible for any natural number to be expressed as a product of 2 or more numbers other than 1 and itself?

Natural numbers, greater than 1 that has no divisors other than 1 and itself are called **prime numbers**.

Even numbers are the numbers divisible by 2 (they have 2 as a divisor), so they can be factorized as 2 times something else. Can an even number be a prime number? Is there any even prime number?

Prime factorization (or integer **factorization)** is a decomposition of a natural number into the product of prime numbers.

Any natural number has single unique prime factorization.

168	2
84	2
42	2
21	3
7	7
1	

180	2
90	2
45	3
15	3
5	5
1	

Prime factorization process:

Prime factors of 168 are $2, 2, 2, 3, 7$, and prime factors of 180 are $2, 2, 3, 3, 5,$

$$2 \cdot 2 \cdot 2 \cdot 3 \cdot 7 = 168; \quad 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 = 180$$

For Halloween, the Jonson family bought 168 mini chocolate bars and 180 gummy worms. What is the largest number of kids among whom the Jonsons can evenly divide both kinds of candy? To solve this problem, we have to find a number that can serve as a divisor for 168 as well as for 180. There are several such numbers. The first one is 2. Both piles of candy can be evenly divided between just 2 kids. 3 is also a divisor. The Jonson family wants to treat as many kids as possible with equal numbers of candy. To do this, they have to find the Greatest Common Divisor (GCD), which is the largest number that can be a divisor for both 168 and 180.

Let’s take a look at the set of all prime factors of 168 and 180. For 168 this set contains 2, 2, 2, 3, and 7. Any of these numbers, as well as any of their products, can be a divisor for 168. The same is true for the set of prime factors of 180, which are 2, 2, 3, 3, and 5. It is easy to see that these two representations have common factors, 2, 2, and 3. This means that both numbers are divisible by any of these common factors and by any of their products. The largest product is the product of all common factors. This largest product has a name: Greatest Common Factor, or GCF. This GCF will also be the Greatest Common Divisor (GCD). $GCF(168, 180) = 12$
They can divide both kinds of candy evenly between 12 kids.

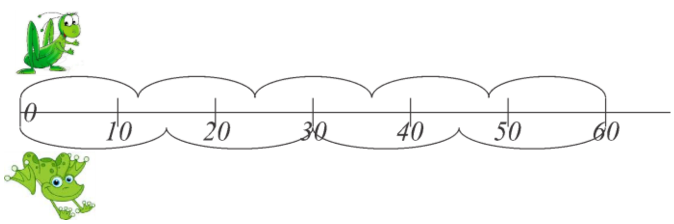
$$2 \cdot 2 \cdot 2 \cdot 3 \cdot 7 = 168$$

$$2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 = 180$$

$$168 \div 12 = 14$$

$$180 \div 12 = 15$$

A grasshopper jumps a distance of 12 centimeters each leap, while a little frog covers a distance of 15 centimeters per leap. They both start at 0 and hop along the long ruler. What is the closest point on the ruler at which they can meet?
There are specific points on the ruler that both of them can reach after a certain number of leaps. One of these points is, of course, 180 cm.
A grasshopper would make 15 jumps, while a frog would make only 12. Will it be the only place where they can meet or are there other possible meeting points?
Any multiple of 180 will also be divisible by 12 and by 15. Are there any other common multiples of 12 and 15, which are less than $12 \cdot 15$ and are still divisible by 12 and 15?



12	2
6	2
3	3
1	

15	3
5	5
1	

Prime factorization of 12 and 15:

$$12 \cdot 15 = (2 \cdot 2 \cdot 3) \cdot (3 \cdot 5)$$

The number which we are looking for has to be a product of prime factors of 12 and 15.

$$2 \cdot 2 \cdot 3 = 12$$

$$3 \cdot 5 = 15$$

$$2 \cdot 2 \cdot 3 \cdot 5 = 60$$

60 is the smallest number, which is divisible by both, 12 and by15, Least Common Multiple (LCM).

$$LCM(12, 15) = 60$$

The Johnson family wants to buy the same number of gammy worms and mini chocolates (168 mini per box and 180 gammy worms per box). How many boxes of each type of candy do they need to buy?

$$7 \cdot 2 \cdot 2 \cdot 2 \cdot 3 = 168$$
$$2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 = 180$$
$$7 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 = 2520$$

If all factors from one number are multiplied by the factors from the second number, which are missing in the first one, the resulting number is divisible by both numbers (is a common multiple) and is the smallest common multiple.

They need to buy $2520 \div 168 = 15$ boxes of mini chocolates and $2520 \div 180 = 14$ boxes of gammy worms. Least common multiple for 168 and180 is 2520, which is much smaller than $168 \cdot 180 = 30240$.

Eratosthenes proposed a simple algorithm for finding prime numbers. This algorithm is known in mathematics as the Sieve of Eratosthenes, a simple, ancient algorithm for finding all prime numbers up to any given limit. It does so by iteratively marking as composite, i.e., not prime, the multiples of each prime, starting with the multiples of 2.

1	2	3	4	5	6	7	8	9
11	12	13	14	15	16	17	18	19
21	22	23	24	25	26	27	28	29
31	32	33	34	35	36	37	38	39
41	42	43	44	45	46	47	48	49
51	52	53	54	55	56	57	58	59
61	62	63	64	65	66	67	68	69
71	72	73	74	75	76	77	78	79
81	82	83	84	85	86	87	88	89
91	92	93	94	95	96	97	98	99

Exercises.

1. Prove that the following numbers are not prime:

   $a.\ 25; \quad b.\ 8192; \quad c. 99.$

2. Do the prime decomposition of the numbers: 66, 28, 128, 555, 1233

3. Find GCF:

   $a.\ GCF(8, 48);$ $b.\ GCF(7,15);$ $c. GCF(20, 1000);$

   $d.\ GCF(23, 69);$ $e.\ GCF(380, 381);$ $f. GCF(14, 25);$

4. Find GCF using prime decomposition:

   $a.\ GCF(75, 135);$ $b.\ GCF(180,210);$ $c. GCF(125, 462);$

   $d.\ GCF(504, 270);$ $e.\ GCF(117, 195, 312);$ $f. GCF(306, 340, 850);$

5. A teacher divided 87 notebooks between the students in the class equally. How many students in the class and how many notebooks did each student get?

6. Mary wrote down a sequence of multiples of a certain number, starting with the smallest one. The twelfth number in this sequence is 60. Find the first, sixth, and twentieth numbers?

7. How many multiples of 9 among first 100 (natural) numbers?
8. Find the LCM using the prime decomposition:
   $a.\ LCM(28, 35)$; $b.\ LCM(16, 56)$; $c. LCM(21, 100)$; $d.\ LCM(18, 62)$;

   $e.\ LCM(264, 300)$; $f. LCM(360, 1020)$; $g.\ LCM(72, 90, 96)$; $h.\ LCM(58, 87, 435)$

9. A florist has 36 roses, 90 lilies, and 60 daisies. What is the largest number of bouquets he can create from these flowers, evenly dividing each kind of flowers between them?

10. There are less than 100 apples in a box. They can be evenly divided between 2, 3, 4, 5, and 6 kids. How many apples are there in the box?

11. Knives are sold 10 to a package, forks are sold 12 to a package, and spoons are sold 15 to a package. If you want to have the same number of each item for a party, what is the least number of packages of each you need to buy?

12. Two buses leave from the same bus station, following two different routes. The first bus takes 48 minutes to complete the round-trip route, while the second one takes 1 hour and 12 minutes to complete the round-trip route. How much time will it take for the buses to meet at the bus station for the first time after they have departed for their routes at the same time?

13. 1In the depot, 3 trains were formed from identical cars. The first one has 418 seats, the second one has 456 seats, and the third one has 494 seats. How many cars are in each train if it is known that the total number of cars does not exceed 50?

14. Mary has a rectangular backyard with sides of 48 and 40 yards. She wants to create square flower beds, all of the equal size, and plant different kinds of flowers in each flower bed. What is the largest possible size of her square flower bed?

15. Solve the problems:
    a. 57 apples were put into boxes, with 6 apples in each box. How many apples are left over?
    b. When peaches were put into 36 boxes, with 12 peaches in each box, 7 peaches were left over. How many peaches were there altogether?

c. There were 120 candies at a party. When each kid took 4 candies, 12 candies were left over. How many kids were at the party?

16. Number 882 is divisible by 147, the number 147 is divisible by 21. Is 28 a divisor of 672?

17. Is number a divisible by number b? if yes, find the quotient.

$a.\ a = 2 \cdot 2 \cdot 3 \cdot 7 \cdot 7, \qquad b = 2 \cdot 2 \cdot 11$

$b.\ a = 2 \cdot 3 \cdot 5 \cdot 13, \qquad b = 5 \cdot 13$

$c.\ a = 3 \cdot 5 \cdot 5 \cdot 11 \cdot 17, \qquad b = 3 \cdot 5 \cdot 17$

$d.\ a = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 19 \cdot 23, \qquad b = 2 \cdot 2 \cdot 3 \cdot 5$

$e.\ a = 2 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 5 \cdot 11 \cdot 13, \qquad b = 405$

$f.\ a = 2 \cdot 3 \cdot 7 \cdot 11 \cdot 13 \cdot 29, \qquad b = 2002$

18. What digit should be placed instead of * so that the number would be divisible by 9:
$a.\ 318*;$ $\quad b.\ *56;$ $\quad c.\ 48*25;$ $\quad d.\ 8*1;$

19. What digit should be placed instead of * so that the number would be divisible by 3:
$a.\ 318*;$ $\quad b.\ *56;$ $\quad c.\ 48*25;$ $\quad d.\ 8*1;$

20. Vinny the Pooh claimed that he knows the number, the product of whose digits is 6552. The Rabbit says it's not true. How does he know?

21. The Johnsons family traveled from Stony Brook to Chicago. They covered the distance between these cities of 846 miles in 3 days. On Friday and Saturday, they covered 620 miles, on Sunday 53 miles more than on Saturday. How many miles did they drive on each of those days?

22. The Johnson family bought tomatoes, cucumbers, and onions on the farm, 18 kg altogether. How many kilograms of each vegetable did they buy if cucumbers were four times as much as onions, and tomatoes were as much as cucumbers?

23. Find missing digits in the problems:

$$\begin{array}{r} 35\square78 \\ +\,4\square596 \\ 678\square \\ \hline 894\square5 \end{array} \qquad \begin{array}{r} 5\square728 \\ +\quad 7045 \\ 83\square50 \\ 821\square\square \\ \hline 227165 \end{array}$$

24. Find the unknown:

25. Replace the addition with multiplication and evaluate:

Example:

$$\underbrace{150+150+\cdots+150}_{20\ times}+\underbrace{20+20+\cdots+20}_{150\ times}=150\cdot 20+20\cdot 150=2\cdot 20\cdot 150=6000$$

$$\underbrace{10+10+\cdots+10}_{101\ times}+\underbrace{10+10+\cdots+10}_{101\ times}$$

26. A sequence of four numbers was recorded, each of which is 3 times larger than the previous one. The last number is 486. Find the first number.

27. A sequence of four numbers was recorded, each of which is 6 times smaller than the previous one. The last number is 2. Find the first number.

28. Find the values of both expressions and compare:

$a.\ 5\cdot(8+14)\ \ and\ 5\cdot 8+14;$ $\qquad b.\ 5\cdot(6+4)\cdot 25\ and\ 5\cdot 6+4\cdot 25$

$c.\ 12+60:(6:2)\ \ and\ \ (12+60):6:2;$ $\qquad d.\ 5\cdot(20-6)+40\ and\ 5\cdot 20-(6+40)$

29. Fill the empty spaces in the magic squares so that all the columns, rows, and diagonals have the same sum of numbers (all numbers should be different).

8		4
		9
	7	

	1	
3		7
	9	