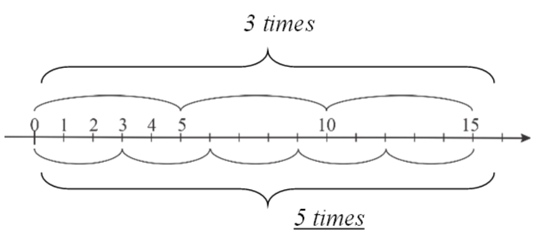
Math 4b. Class work 3(b).

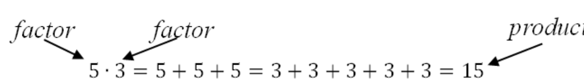
Multiplication and division.

In Chapter 1, we discussed a few properties of addition and multiplication. As we all know, multiplication is an arithmetic operation, equivalent to the repeated addition of the same number.

$$c \cdot b = \underbrace{c + c + \cdots + c}_{b\ times} = \underbrace{b + b + \cdots + b}_{c\ times} = a$$

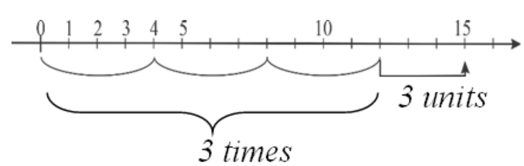
The result of multiplication is called the product, and the numbers involved in the operation are called factors. c and b are factors, and a is a product.
Multiplication is closely connected with division; when we divide a number (this number is called the dividend) by a divisor, we are seeking a number (a quotient) that, when multiplied by the divisor, gives us the dividend.
(In this part of our course, we are discussing natural numbers, which are used for counting and start from 1: 1, 2, 3, and so on. I will omit the word 'natural' and use only the term 'number' for a while.)
If there is a number c, that $c \cdot b = a$, then we can say that $a \div b = c$. This means that a is divisible by b, and b can be "fit" into a a whole number of times. c also is a factor of a, $a: c = b$. For example,





$3 \cdot 5 = 15; \quad 15: 3 = 5, \quad 15: 5 = 3$
5 can fit into 15 exactly 3 times, 3 can go into 15 exactly 5 times.
15 is divisible by 3 and by 5.

If there is no number such that the divisor enters the dividend several times, then we can say that this number is not divisible by the divisor. In such cases, we can use division with a remainder. For example, consider 15: 4. 4 can't fully complete 15. It can fit into 12 three times, but there will be a little more left over. So,



$$15 \div 4 = 3\ with\ a\ remainder\ of\ 3$$

$15: 4 = 3R(3)$, or

$$15 = 4 \cdot 3 + 3$$

For division of any natural number by another, we can now write:

$$a: b = cR(r), \qquad or \qquad a = b: c + r$$

If $r = 0$, number a is divisible by number b.

Why can't we divide by 0? By definition, multiplying 0 by anything results in 0. Dividing by 0 would imply that there is a number that, when multiplied by 0, does not yield 0. But this is impossible, therefore, division by 0 is undefined; it simply does not exist, and we cannot perform such an operation!

Since division of natural numbers by 2 results in two different remainders, the set of all natural numbers can be divided into two classes, each containing infinitely many numbers. The first class includes all numbers that have a remainder of 0 when divided by 2. Here are the first 10 numbers in this class: 2, 4, 6, 8, 10, 12, 14, 16, 18, 20.

The second class includes all numbers that have a remainder of 1 when divided by 2. Here are the first 10 numbers in this class: 1, 3, 5, 7, 9, 11, 13, 15, 17, 19."

The numbers in the first class we call ***even numbers***, and the number in the second — ***odd numbers***.

Divisibility rules.

Can we predict whether a given number is divisible by 2, 3, 4, and so on? Yes, we can do it, based on divisibility rules:

- Any (natural) number is divisible by 1 and by itself.
- A number is divisible by 2 **if and only if** its last digit is even or 0.
- A number is divisible by 3 **if and only if** sum of its digits is divisible by 3.
- A number is divisible by 4 **if and only if** the number formed by the last 2 digits is divisible by 4.
- A number is divisible by 5 **if and only if** its last digit is 5 or 0.
- What can you say about the divisibility rule for division by 6? Write it here:
- A number is divisible by 7 **if and only if** the result of subtracting twice the last digit from the remaining part of the number is also divisible by 7.
- A number is divisible by 8 **if and only if** the number formed by the last 3 digits is divisible by 8.
- A number is divisible by 9 **if and only if** the sum of its digits is divisible by 9.
- What can you say about the divisibility rule for division by 10?
- Number is divisible by 11 **if and only if** the result of alternation addition and subtraction is divisible by 11.

  Example:

  Is number 517 divisible by 11?

  $5 - 1 + 7 = 11.$ 11 is divisible by 11, so 517 is also divisible.

Did you notice the expression "**if and only if**" in the rules of divisibility? What is the difference between these two statements:

A number is divisible by 2 ***if*** *its last digit is even or 0.*
A number is divisible by 2 ***if and only if*** *its last digit is even or 0.*

In the first case, the statement is telling us, that if the number ends with even digit or zero, it is definitely even, but in the case that the last digit is not even or zero, nothing can be said for sure, the number may be divisible by 2, or maybe not. The second statemen is more precise, it defines the outcome.
These rules are true for any number, even very big ones, and all of them can be proved as a true statement. The concept of "*proof* "was not always known in mathematics. First, people just used examples, to show, that some statement is true.
Example:
Even or odd number will be the sum of two odd numbers?
Intuitively, we want to find a few examples and check:
$1 + 5 = 6;\qquad 25 + 29 = 54;\qquad 123 + 277 = 300$
It looks like we are getting an even number when adding two odd numbers. Can we be absolutely sure, that the sum of $\underbrace{12345231\ldots.34567891}_{1000\ digits}$ and $\underbrace{19845672\ldots.53142377}_{1000\ digits}$ is an even number?
Let's reason like this: any odd number is an even number plus 1, and any even number has a factor 2 (it's divisible by 2 by definition, we call the numbers, divisible by 2 "even" numbers). It means that an even number is a product of 2 and some other number.

$$even\ number = 2 \cdot some\ number$$

If two odd numbers are added together

$$2 \cdot some\ number + 1 + 2 \cdot another\ number + 1 =$$
$$2 \cdot some\ number + 2 \cdot another\ number + 2$$

All three terms in this expression have a factor 2, therefore we can use the distributive property and write it as

$$2 \cdot (some\ number + another\ number + 1)$$

And this, actually unknown number, the sum of two random odd numbers has a factor 2, so it is even.
We proved that the sum of **any** two odd numbers is even.
In a very similar way, divisibility rules can be proven, not always in a straightforward way. Some of them we will prove later in our class, some other proves you will learn in the higher grades.

Exercises.

1. Factorize (represent as a product of two or more factors, do not use 1 as a factor). Write one or more possible solutions:
   Example: $35 = 3 \cdot 7;\qquad 100 = 4 \cdot 5 \cdot 5;$
   $$36;\quad 100;\quad 125;\quad 178;\quad 200.$$

2. Do the division, write your answer in a form $a{:}\,b = cR(r)$.

Examples:
$25:4 = 6R(1);$ $28:7 = 4R(0)$
$a.$ $36:5;$ $b.$ $43:4;$ $c.$ $75:3;$ $d.$ $126:5;$ $81:9;$

3. What is the largest remainder that can be obtained when dividing natural numbers:
$a.$ $by\ 2;$ $b.$ $by\ 3;$ $c.$ $by\ 4;$ $d.$ $by\ 5?$

4. If we want to divide a number by 7, what numbers can we get as a remainder? What numbers can we get as remainder by division by11?

5. What is the smallest remainder can we received by division one number by another (excluding 0)?

6. The remainder of $1932:17$ is 11, the remainder of $261:17$ is 6. Is

$$2193 = 1932 + 261$$

divisible by 17? Is it possible to find out without division?

7. Find all natural numbers such that when divided by 5, the quotient and remainder are equal?

8. All natural numbers can be divided into two classes based on their remainder when divided by 2. Into how many classes can numbers be divided based on their remainder when divided by 3? By 4?
9. Factor out the common factor, find the value of the expressions:
Example:

$$21 + 49 = 3 \cdot 7 + 7 \cdot 7 = 7 \cdot (3 + 7) = 7 \cdot 10 = 70$$

$a.$ $35 - 25;$ $b.$ $44 + 77;$ $c.$ $81 - 45;$ $d.$ $56 - 35;$

10. Will the sum and the product be even or odd for:
    a. 2 odd numbers
    b. 2 even numbers
    c. 1 even and 1 odd number
    d. 1 odd and 1 even number

Try do not just provide a few examples, but explain your results.

11. Fill the empty boxes in the table (draw the table in your notebook):

a	20	24	24		77		0	0
b	5	6		12		8	25	

$a \cdot b$			192	576				0
$a : b$					7	9		

12. Find the unknown number:

$a.\ 18 \cdot x = 450;$ $b.\ 1190 : c = 34$ $c.\ 25 \cdot x = 1000$

$d.\ d \cdot 23 = 2346;$ $e.\ n : 17 = 22$ $f.\ 37 \cdot x = 851$

13. Calculate by grouping the identical terms:

$a.\ 9 + 5 + 5 + 9 + 9 + 5 + 9 + 5 + 9 + 5 + 9 + 5 + 5;$

$b.\ 6 + 3 + 6 + 2 + 3 + 2 + 6 + 2 + 3 + 3 + 2 + 2.$

14. Evaluate:

$a.\ 3 \cdot 5 \cdot 2 \cdot 7;$ $b.\ 5 \cdot 5 \cdot 6 \cdot 4;$ $c.\ 7 \cdot 2 \cdot 5 \cdot 2 \cdot 5$

$d.\ 2 \cdot 9 \cdot 5 \cdot 5 \cdot 4;$ $e.\ 8 \cdot 4 \cdot 125 \cdot$ $c.\ 5 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 5 \cdot 5 \cdot 5 \cdot 6;$

15. It is known, that $x \cdot y = 12$ What is the value of:

$a.\ x \cdot (y \cdot 5);$ $b.\ (x \cdot 2) \cdot y;$ $c.\ y \cdot (x \cdot 10);$ $d.\ (y \cdot 2) \cdot (x \cdot 3);$

16. Write all divisors of numbers: 8, 12, 15, 36

Example: $D(8)\ are\ 1, 2, 4, 8$

17. Is the product of 1247 and 999 divisible by 3 (no calculations)?

18. Number a is divisible by 5. Is the product $a \cdot b$ divisible by 5? Explain.

19. Without calculating, establish whether the product is divisible by a number?

$a.\ 508 \cdot 12\ by\ 3$ $b.\ 85 \cdot 3719\ by\ 5$

$c.\ 2510 \cdot 74\ by\ 37$ $d.\ 45 \cdot 26 \cdot 36\ by\ 15$

$e.\ 210 \cdot 29\ by\ 3, by\ 29$ $f.\ 3800 \cdot 44 \cdot 18\ by\ 11, 100, 9$

20. Without calculating, establish whether the sum is divisible by a number:

$a.\ 25 + 35 + 15 + 45 \quad by\ 5;$ $b.\ 14 + 21 + 63 + 24\ by\ 7$

$c.\ 18 + 36 + 55 + 90 \quad by\ 9;$