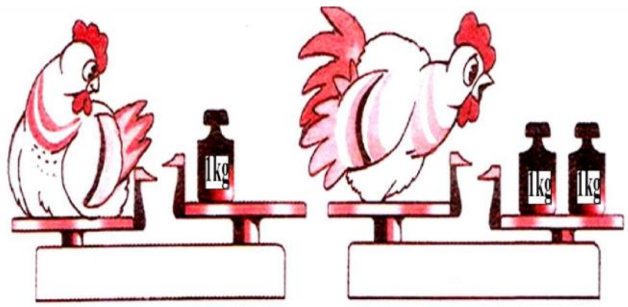
Math 4b. Class work 11.

Decimals

In a process of measurement, we compare a standard unit, such as 1m for length, 1kg for mass, 1degree Celsius for temperature, and so on (we can use another standard units, for example 1 foot, 1 degree Fahrenheit) with the quantity we are measuring. It is very likely that our measurement will not be exact and whole

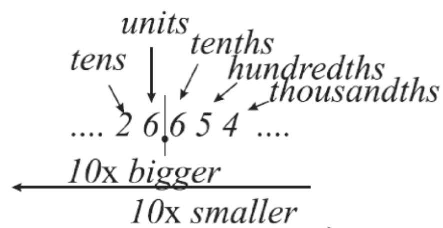


number of standard units will be either smaller, or greater than the measured quantity. In order to carry out more accurate measurement, we have to break our standard unit into smaller equal parts. We can do this in many different ways. For example, we can take $\frac{1}{2}$ of a standard unit and continue measuring. If we didn't get exact n units plus $\frac{1}{2}$ of a unit, we have to subdivide further:

$$n+\frac{1}{2}+\frac{1}{2}\cdot\left(\frac{1}{2}\right)+\cdots$$

It turns out that perhaps the most convenient way is to divide a unit into 10 equal parts, then each of one tenth into another 10 even smaller equal parts and so on. In this way we will get a series of fractions with denominators 10, 100, 1000 and so on:

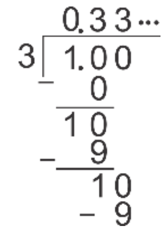
$$\frac{1}{10},\frac{1}{100},\frac{1}{1000}\ldots$$



The result of our measurement can be written in a 10 based place value system.

$$26.654=10\cdot 2+1\cdot 6+\frac{1}{10}\cdot 6+\frac{1}{100}\cdot 5+\frac{1}{1000}\cdot 4$$
$$=10\cdot 2+1\cdot 6+\frac{6}{10}\cdot+\frac{5}{100}+\frac{4}{1000}$$
$$=10\cdot 2+1\cdot 6+\frac{600}{1000}+\frac{50}{1000}+\frac{4}{1000}$$

Of course, all such numbers can be expressed in the fractional notation as fractions with denominators 10, 100, 1000 …, but in decimal notation all arithmetic operations are much easier to perform.



How the fraction can be represented as decimal? One way to do it, just divide numerator by denominator, as usual. For example:

$$\frac{1}{3} = 1:3 = 0.3333\ldots = 0.\overline{3}$$

Another example,

$$\frac{2}{11} = 2:11 = 0.1818\ldots = 0.\overline{18}$$

$$\frac{3}{5} = 3:5 = 0.6$$

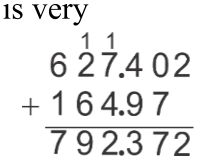
Can you notice the difference? If the denominator of the fraction can be prime factorized into the product of only 2 and/or 5, the fraction can be written as a fraction with denominator 10, 100, 1000 … Such fraction can be represented as a finite decimal; any other fraction will be written as infinite periodical decimal or repeating decimal. Set of digits which repeats in a repeating decimal is called repetend. For now, we are going to work only with finite decimals.
Examples:

$$0.3 = \frac{3}{10}; \qquad 0.27 = \frac{2}{10} + \frac{7}{100} = \frac{27}{100}; \qquad 0.75 = \frac{75}{100} = \frac{3 \cdot 25}{4 \cdot 25} = \frac{3}{4}$$

$$\frac{1}{25} = \frac{1}{5 \cdot 5} = \frac{1 \cdot 2 \cdot 2}{5 \cdot 5 \cdot 2 \cdot 2} = \frac{4}{10 \cdot 10} = \frac{4}{100} = 0.04$$

$$\frac{7}{8} = \frac{7}{2 \cdot 2 \cdot 2} = \frac{7 \cdot 5 \cdot 5 \cdot 5}{2 \cdot 2 \cdot 2 \cdot 5 \cdot 5 \cdot 5} = \frac{875}{1000} = 0.875$$

As you probably already noticed, the decimal (based on 10) system of writing numbers is very consistent, and we can write very big and very small numbers in a very similar way. This system is very convenient when we perform the arithmetic calculations. Addition and subtraction can be completed by exactly the same way as with natural numbers, decimal point should be placed one on the top of the other.



How do we perform the multiplication? First, let's see why we need to write 0 on the right side of the number when multiplying it by 10:

$$245 \cdot 10 = (100 \cdot 2 + 10 \cdot 4 + 5) \cdot 10 = 100 \cdot 10 \cdot 2 + 10 \cdot 10 \cdot 4 + 10 \cdot 5$$
$$= 1000 \cdot 2 + 100 \cdot 4 + 10 \cdot 5 + 1 \cdot 0 = 2450$$

By multiplying the number by 10 we change the place values for all digits and added one more place for units.
If the number has a fractional decimal part, and is multiplied by 10:

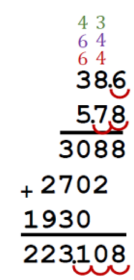
$$45.34 \cdot 10 = \left(10 \cdot 4 + 5 + \frac{1}{10} \cdot 3 + \frac{1}{100} \cdot 4\right) \cdot 10$$
$$= 10 \cdot 10 \cdot 4 + 10 \cdot 5 + \frac{1}{10} \cdot 10 \cdot 3 + \frac{1}{100} \cdot 10 \cdot 4$$
$$= 100 \cdot 4 + 10 \cdot 5 + 3 + \frac{1}{10} \cdot 4 = 453.4$$

Value of each place in the number was increased 10 times; decimal point moved one step to the right
(2 steps for multiplication by 100, and so on),
To divide a number by 10, we just multiply it by $\frac{1}{10}$.

$$230: 10 = 230 \cdot \frac{1}{10} = (100 \cdot 2 + 10 \cdot 3 + 1 \cdot 0) \cdot \frac{1}{10} = \frac{100}{10} \cdot 2 + \frac{10}{10} \cdot 3 + \frac{0}{10} = 10 \cdot 2 + 3$$
$$= 23$$

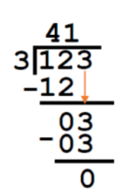
$$235: 10 = 235 \cdot \frac{1}{10} = (100 \cdot 2 + 10 \cdot 3 + 1 \cdot 5) \cdot \frac{1}{10} = \frac{100}{10} \cdot 2 + \frac{10}{10} \cdot 3 + \frac{1}{10} \cdot 5$$
$$= 10 \cdot 2 + 3 + \frac{1}{10} \cdot 5 = 23.5$$

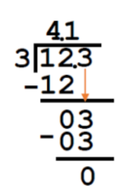
As the result, we are reducing values of each place 10 times, and we have to move the decimal point one step to the left.
To perform the long multiplication of the decimals, we do the multiplication procedure as we would do with natural numbers, regardless the position of decimal points, then the decimal point should be placed on the resulting line as many steps from the right side as the sum of decimal digits of all factors. When we did the multiplication, we didn't take into the consideration the fact, that we are working with decimals, it is equivalent to the multiplication of each number by 10 or 100 or 1000 … (depends on how many decimal digits it has). Therefore, the result we got is greater by $10 \cdot 100 = 1000$ (in our example) time than the one we are looking for:

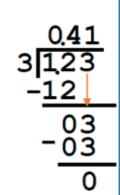


$$38.6 \cdot 5.78 = 38.6 \cdot 10 \cdot 5.78 \cdot 100: (10 \cdot 100) = 386 \cdot 578: 1000$$

Division. To do the long division of the decimal by a hole number, do it as usual, but when the first digit after the decimal point is going to be placed down, put the decimal point into the answer.







$$12.3:3 = \left(10 \cdot 1 + 2 + \frac{1}{10} \cdot 3\right):3 = \left(10 \cdot 1 + 2 + \frac{1}{10} \cdot 3\right) \cdot \frac{1}{3} = 12 \cdot \frac{1}{3} + \frac{1}{10} \cdot 3 \cdot \frac{1}{3} = 4 + \frac{1}{10} = 4.1$$

$$1.23:3 = \left(1 + \frac{1}{10} \cdot 2 + \frac{1}{100} \cdot 3\right):3 = \left(1 + \frac{1}{10} \cdot 2 + \frac{1}{100} \cdot 3\right) \cdot \frac{1}{3} = \frac{12}{10} \cdot \frac{1}{3} + \frac{1}{100} \cdot 3 \cdot \frac{1}{3} = \frac{4}{10} + \frac{1}{100} = 0.4 + 0.01 = 0.41$$

If the divisor is decimal as well, it needs to be multiplied by 10(100, 1000…), depends on how many digits it has after decimal point; the dividend also should be multiplied by the same number to get the correct answer. (If we are dividing by a number which is 10 times larger, the number to be divided is also should be 10 times bigger. For example, to divide 123.452 by 1.23 we need to multiply 1.23 by 100:

$$1.23 \cdot 100 = 123$$

to get a whole number. Then multiply123.452 by 100 too.

$$123.452 \cdot 100 = 12345.2$$

Exercises:

1. Write in decimal notation the following fractions:

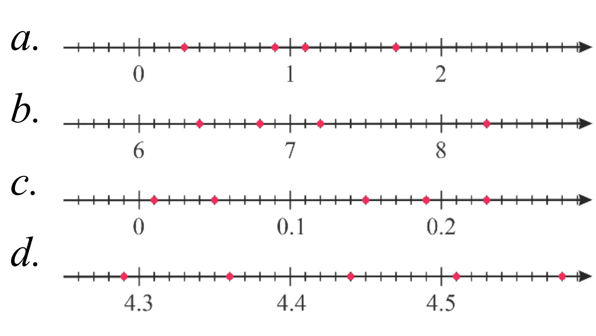
Example:

$$1\frac{3}{25} = 1 + \frac{3}{25} = 1 + \frac{3 \cdot 4}{25 \cdot 4} = 1 + \frac{12}{100} = 1.12$$

$a.\ 1\frac{1}{10};\quad 2\frac{4}{10};\quad 4\frac{9}{10};\quad 24\frac{25}{100};\quad 98\frac{3}{100};$

$b.\ 1\frac{1}{100};\quad 4\frac{333}{1000};\quad 8\frac{45}{1000};\quad 75\frac{8}{10000};\quad 9\frac{565}{10000}$

2. Which numbers are marked on the number lines:



3. Evaluate:

$a.\ 1.2 + 2.3 + 3.4 + 4.5 + 5.6 + 6.7 + 7.8;$

$b.\ 2.3+3.4+4.5-5.6+6.7+7.8+8.5+9.2;$

$c.\ 1.7+3.3+7.72+3.28+1.11+8.89;$

$d.\ 18.8+19+12.2+11.4+0.6+11;$

4. Evaluate:

   $a.\ 2-0.7-0.04;$ $b.\ 4-0.4-0.05;$

   $c.\ 3-(1-0.22);$ $d.\ (5-0.1)-(1-0.2)$

5. On a graph paper draw a number line, use 10 squares as a unit. Mark points with coordinates 0.1, 0.5, 0.7, 1.2, 1.3, 1.9.

6. Write the numbers in an extended form;

   Example:

   $312.23 = 100\cdot 3 + 10\cdot 1 + 1\cdot 2 + \frac{1}{10}\cdot 2 + \frac{1}{100}\cdot 3$

   34.2; 231.51; 76.243; 25.34; 0.23; 0.0023

7. Write in decimal notation:

   $\frac{173}{10};$ $\frac{173}{100};$ $\frac{173}{1000};$ $\frac{173}{1000};$

8. Write in decimal notation:

   $a.\ 2\frac{18}{100};$ $5\frac{3}{100};$ $1\frac{238}{1000};$ $8\frac{8}{1000};$ $b.\ \frac{39}{10};$ $\frac{187}{10};$ $\frac{341}{100};$ $\frac{1002}{1000}$

9. Which fractions below can be written in as a finite decimal:

   $\frac{1}{2}, \frac{2}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{7}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10},$

   $\frac{1}{11}, \frac{1}{12}, \frac{1}{13}, \frac{1}{14}, \frac{1}{15}, \frac{1}{16}, \frac{1}{17}$

   Why do you think so?

10. Write decimals as fractions and evaluate the following expressions:

    $a.\ \frac{2}{3}+0.5;$ $b.\ \frac{1}{3}\cdot 0.9;$ $c.\ \frac{3}{16}\cdot 0.16$

    $d.\ 0.6-\frac{2;}{5}$ $e.\ 0.4:\frac{2}{7};$ $f.\ \frac{9}{20}:0.03$