MATH CLUB
ASSIGNMENT 6: INCLUSION–EXCLUSION FORMULA
NOVEMBER 23, 2025

Combinatorics – inclusion-exclusion formula

It is well known that if we have two finite sets A, B, then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

(where $|A|$ is the number of elements in A).

One can generalise it to three sets:

$$\begin{aligned} |A \cup B \cup C| = &|A| + |B| + |C| \\ &- |A \cap B| - |A \cap C| - |B \cap C| \\ &+ |A \cap B \cap C| \end{aligned}$$

It turns out that there is a similar formula for any number of sets:

$$\begin{aligned} |A_1 \cup A_2 \cup \dots \cup A_n| = &|A_1| + \dots + |A_n| \\ &- |A_1 \cap A_2| - \dots \qquad \text{(all possible pairwise intersections)} \\ &+ |A_1 \cap A_2 \cap A_3| + \dots \qquad \text{(all possible triple intersections)} \\ &\dots \end{aligned}$$

This formula is called *inclusion-exclusion formula*

1. (a) Prove inclusion-exclusion formula for 3 sets.
(b) Prove inclusion-exclusion formula for n sets. [Hint: if an element is exactly one of these sets, how many times will it be counted in the right-hand side of inclusion-exclusion formula? what if it is in exactly 2 sets?...]

2. How many integers are there between 1 and 16500 which are not divisible by neither 3 nor 5? What about integers which are not divisible by either of 3, 5, 11?

3. How many ways there are to put 15 different toys in 4 boxes so that each box contains at least one toy? [Hint: let A_i, $i = 1 \dots 4$, be the set of those arrangements where box i is empty. Then $A_1 \cup A_2 \cup A_3 \cup A_4$ are those arrangements where at least one box is empty.]

4. Inclusion-exclusion formula has an analog where finite sets are replaced by figures in the plane and number of elements is replaced by area (we assume that all figures are nice enough so that notion of area makes sense and all areas are finite). State and prove this analog.

5. Inside a rectangle of area 1 we have 5 figures $F_1, \dots, F_5$, each of which has area 0.5. Prove the following:
(a) You can choose two of these figures so that their intersection has area at least $3/20$.
(b) You can choose two of these figures so that their intersection has area at least $1/5$.
(c) You can choose three of these figures so that their intersection has area at least $1/20$.

6. A professor wrote 30 letters to his colleagues, and prepared 30 envelopes with their addresses. However, he was very much preoccupied with a theorem he was trying to prove and was not paying much attention to anything else, so he put the letters in the envelopes at random.

What is the probability that at least one letter was placed in the correct envelope?

[This is a famous problem which appears in many forms; it seems that it was first proposed in 1708 by a French mathematician Pierre Rémond de Montmort who called it “the problem of coincidences”. A highly unexpected result is that if we replace 30 by n and ask how the probability behaves as n grows, it approaches a limit – and this limit is neither 0 nor 1. Many famous mathematicians worked on questions related to this problem, including Bernoulli, de Moivre, and Euler.]

***7.** (For those who have solved all previous problems)

We color all integer numbers between 1 and 1 000 000 using one of two colors, black and white. Initially all numbers are black. You are allowed to perform the following operation: chose a number n (between 1 and 1 000 000) and switch colors of n and all integers that have a common divisor > 1 with n.

Prove that using this operation, you can make all numbers white.