MATH CLUB 2025
AMC PREPARATION: NUMBER THEORY
OCT 26, 2025

Facts / Theorems

A lot of the exercises will require some number theory results. It would likely be helpful to go through the following list before starting the exercises:

- LCM and how to find it from prime factorization
- GCD and how to find it from prime factorization
- Euclidean algorithm
- Counting divisors
- Modular arithmetic rules
- Modular inverses (using Euclidean algorithm)
- Difference of squares and sum/difference of cubes factorization
- Fermat's Little Theorem
- Chinese Remainder Theorem

Additional facts to recall:

1. **Bezout's Lemma:** Given two positive integers a, b, the smallest positive integer that can be written as $ma + nb$, where m, n are any integers, is $\gcd(a, b)$.
2. If $\gcd(a, b) = 1$, then multiplication by b doesn't affect divisibility by a: $a|bx \iff a|x$.
3. **Representing numbers in different bases:** A base-10 number $abcd$ means
$$a \cdot 10^3 + b \cdot 10^2 + c \cdot 10^1 + d \cdot 10^0.$$
In base B, $wxyz_B$ means
$$w \cdot B^3 + x \cdot B^2 + y \cdot B^1 + z \cdot B^0.$$
4. **Euler's Theorem:** If $\gcd(a, m) = 1$, then
$$a^{\varphi(m)} \equiv 1 \pmod{m},$$
where $\varphi(m)$ (Euler's Totient Function) is the number of positive integers less than m that are relatively prime to m. If
$$m = p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n},$$
then
$$\varphi(m) = m\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_n}\right).$$

Exercises

1. **(2020 AMC 8 #17)** How many positive integer factors of 2020 have more than 3 factors?
2. **(2022 AMC 10A #7)** The least common multiple of a positive integer n and 18 is 180, and the greatest common divisor of n and 45 is 15. What is n?
3. **(1990 AHSME #11)** How many positive integers less than 50 have an odd number of positive integer divisors?
4. **(2006 AMC 10B #11)** What is the tens digit in the sum $7! + 8! + 9! + \cdots + 2006!$?
5. **(2005 AMC 10A #14)** How many three-digit numbers satisfy the property that the middle digit is the average of the first and last digits?
6. **(2005 AMC 10A #15)** How many positive cubes divide $3! \cdot 5! \cdot 7!$?
7. **(2003 AMC 10A #16)** What is the units digit of 13^{2003}?
8. **(2007 AMC 10A #17)** Suppose m and n are positive integers such that $75m = n^3$. What is the minimum possible value of $m + n$?

9. **(2015 AMC 10A #18)** Among the first 1000 positive integers, there are n whose hexadecimal representation contains only numeric digits. What is the sum of the digits of n?

10. **(2018 AMC 10A #19)** A number m is randomly selected from $\{11, 13, 15, 17, 19\}$, and a number n is randomly selected from $\{1999, 2000, \ldots, 2018\}$. What is the probability that m^n has a units digit of 1?

11. **(2003 AMC 10A #20)** How many base-10 three-digit numbers n also have a three-digit base-9 and base-11 representation?

12. **(2005 AMC 10A #21)** For how many positive integers n does $1 + 2 + \cdots + n$ evenly divide $6n$?

13. **(2005 AMC 10A #22)** Let S be the set of the 2005 smallest positive multiples of 4, and let T be the set of the 2005 smallest positive multiples of 6. How many elements are common to S and T?

14. **(2016 AMC 10A #22)** For some positive integer n, the number $110n^3$ has 110 positive integer divisors. How many divisors does $81n^4$ have?

15. **(2007 AMC 10A #23)** How many ordered pairs (m, n) of positive integers with $m \geq n$ have the property that their squares differ by 96?

16. **(2005 AMC 10B #24)** Let x and y be two-digit integers such that y is obtained by reversing the digits of x. The integers satisfy $x^2 - y^2 = m^2$ for some m. What is $x + y + m$?

17. **(2006 AMC 10B #25)** Mr. Jones has eight children of different ages. His oldest child, who is 9, spots a license plate with a 4-digit number in which each of two digits appears twice. "Look, Daddy!" she exclaims. "That number is evenly divisible by the age of each of us kids!" "That's right," replies Mr. Jones, "and the last two digits just happen to be my age." Which of the following is *not* one of the children's ages?

    (A) 4 (B) 5 (C) 6 (D) 7 (E) 8

18. **(2002 AMC 12B #11)** The positive integers $A, B, A - B$, and $A + B$ are all prime. The sum of these four primes is

    (A) even (B) divisible by 3 (C) divisible by 5 (D) divisible by 7 (E) prime

19. **(2002 AMC 12B #12)** For how many integers n is $\frac{n}{20-n}$ the square of an integer?

20. **(2007 AMC 12A #12)** Integers a, b, c, d are chosen from 0 to 2007. What is the probability that $ad - bc$ is even?

21. **(2003 AMC 12A #12)** Sally has five red cards numbered 1–5 and four blue cards numbered 3–6. She stacks them so that colors alternate and each red number divides both neighboring blue numbers. What is the sum of the middle three cards?

22. **(2006 AMC 12A #14)** Two farmers agree that pigs are worth 300 dollars and goats 210 dollars. When paying debts, they exchange pigs and goats as necessary. What is the smallest positive debt that can be resolved in this way?

23. **(2011 AMC 12B #15)** How many positive two-digit integers are factors of $2^{24} - 1$?

24. **(2009 AMC 12A #18)** For $k > 0$, let $I_k = 10\ldots064$, with k zeros between 1 and 6. Let $N(k)$ be the number of factors of 2 in the factorization of I_k. What is the maximum $N(k)$?

25. **(2016 AMC 12B #22)** For a positive integer $n < 1000$, the decimal equivalent of $\frac{1}{n}$ has period 6, and $\frac{1}{n+6}$ has period 4. In which interval does n lie?

    (A) [1,200] (B) [201,400] (C) [401,600] (D) [601,800] (E) [801,999]

26. **(2010 AMC 12A #23)** The number obtained from the last two nonzero digits of 90! is equal to n. What is n?