

MATH CLUB: POLYNOMIALS AND ROOTS

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SOME USEFUL FACTS ABOUT POLYNOMIALS

- **Long division:** given polynomials $f(x)$, $g(x)$ (with degree of $g(x)$ at least 1), one can uniquely write $f(x)$ in the form

$$f(x) = q(x)g(x) + r(x), \quad \deg r(x) < \deg g(x)$$

Polynomials $q(x)$, $r(x)$ are called *quotient* and *remainder* respectively.

- **Bezout theorem:** when a polynomial $P(x)$ is divided by $(x-c)$, the remainder is $P(c)$. In particular, $P(x)$ is divisible by $(x-c)$ if and only if c is a root, i.e. $P(c) = 0$.

Moreover, if $P(x)$ has integer coefficients and c is an integer root, then $P(x)$ is divisible by $(x-c)$ and the quotient has integer coefficients.

PROBLEMS

1. Find the remainder when $x^{13} + 1$ is divided by $x - 1$
2. The polynomial $P(x)$ has remainder 99 when divided by $x - 19$ and remainder 19 when divided by $x - 99$. What is the remainder when $P(x)$ is divided by $(x - 19)(x - 99)$?
3. Let $P(x)$ be a polynomial with integer coefficients and let a, b be integers, $a \neq b$. Prove that then $P(a) - P(b)$ is divisible by $(a - b)$.
4. Is it possible to find a polynomial with integer coefficients such that $P(7) = 11$ and $P(11) = 13$?
5. Prove that $x^{2n} + x^n + 1$ is divisible by $x^2 + x + 1$ if and only if n is not a multiple of 3.
6. Find the remainder when $x^{81} + x^{49} + x^{25} + x^9 + x$ is divided by $x^3 - x$.
7. Is it true that if the polynomial $P(x)$ is such that $P(n)$ is an integer for any integer n , then $P(x)$ has integer coefficients?
8. Construct a quadratic polynomial $f(x)$ such that $f(-1) = 1$, $f(0) = 0$, $f(2) = 4$.
9. A ship is traveling at constant speed keeping the same course without turning. The captain is measuring the distance to remote lighthouse every hour.
 - At noon, the distance was 10 miles.
 - At 3 pm, the distance was $10\sqrt{2}$ miles
 - At 5pm, the distance was $10\sqrt{10}$ miles.
 - What will the distance be at midnight?
- *10. Does there exist a polynomial with integer coefficients $P(x)$ such that for every integer n , $P(n)$ is a prime number?