MATH CLUB ASSIGNMENT 1: PIGEONHOLE PRINCIPLE

SEP 21, 2025

Welcome to the new semester at SchoolNova!!

The math club is where we solve challenging but enjoyable problems from all areas of math — usually the problems that are not part of the standard syllabus.

We will participate in some olympiads, and these problems will help you get ready for these competitions. However, the primary goal of the class is enjoying challenging math, not olympiad preparation — these two goals are close, but not exactly the same.

I ask that each student bring a notebook (preferably quad ruled), pencils and a folder or binder to keep old assignments — you will need them!

If you have any questions, please contact me by email: kirillov@schoolnova.org.

You might have seen some of these problems before. If so, let me know, and I will give you other problems to work on.

The pigeonhole principle

If you put n items in m boxes, with $n > m$, then at least one box will have more than one item.

Generalization
If $n > km$ objects are put in m boxes, then at least one box will have more than k objects.

Problems

1. (a) Given 3 distinct points with integer coordinates on the real line, prove that one can always choose 2 of them so that the midpoint of the segment connecting them also has integer coordinate.
(b) Same question, but for 5 points in the plane, and we require the midpoint of the segment to have both coordinates integer.
(c) How many points would one need in n-dimensional space?

2. Consider the sequence of numbers 1, 11, 111, 1111, ...,
(a) Prove that among these numbers, there are two whose difference is divisible by 179
*(b) Prove that one of these numbers is divisible by 179.

3. Let A be any set of 19 distinct integers chosen from the arithmetic progression 1, 4, 7, ..., 100. Prove that there must be two distinct integers in A whose sum is 104.

4. Nine points (all distinct, no three on the same line) are placed inside a square with side length 2. Show that one can choose 3 of these points which form a triangle of area $\leq \frac{1}{2}$.

5. Compute (by hand, using long division — it is important!) fractions $\frac{1}{7}$, $\frac{2}{7}$, $\frac{3}{7}$ as infinite decimals. Do you see any patterns?

6. Consider a sequence $a_1, a_2, a_3, \ldots$ which is formed by the following rule: each next term a_{k+1} is obtained by multiplying a_k by 10 and then taking remainder upon division by 7. [Starting term a_1 is chosen arbitrarily.] Show that this will always produce a periodic sequence. What is the maximal period? What happens if instead of 7 we used another number, such as 11 or 12?

7. (a) Explain the relation between the two previous problems.
(b) Argue that any rational number p/q, when written in decimal, is periodic. What is the maximal period?

8. Prove that from a set of ten distinct two-digit numbers (in the decimal system), it is possible to select two disjoint non-empty subsets whose members have the same sum.

[This problem is from 1972 International Math Olympiad, but it is one of the simplest IMO problems. As a hint, try first finding two different such subsets without requiring that they be disjoint.]

9. Given any $n+1$ integers between 1 and $2n$, show that one of them is divisible by another. Is this best possible, i.e., is the conclusion still true for n integers between 1 and $2n$?

Some miscellaneous problems

1. During daytime, a snail climbs 10cm up a post. During the night, it slides down 9cm. How long will it take the snail to reach the top of the pole if the height of the pole is 1m?

2. A rectangular bar of chocolate consists of $m \times n$ squares. You want to break it into mn individual squares. At each step, you may pick up one piece you have and break it along any of the vertical or horizontal lines separating the squares.

How many breaks do you need? What is the fastest way to do it?

3. Alexander has a bucket of wine and bucket of water. He draws 1 quart of wine from the wine bucket and pours it into the bucket with water, carefully stirs it and then draws 1 quart of the mix and pours it into the wine bucket.

What is larger: amount of wine in the water bucket or amount of water in the wine bucket?

4. You have two identical glass balls. Your goal is to find the maximal height from which these balls can be dropped without breaking. To do that, you can drop the balls (one at a a time) from any floor of a 100-story building.

How many attempts will you need?

5. Solve the equation

$$2025 - 2(2025 - 2(2025 - 2(2025 - 2x))) = x$$