Homework 23

Bernoulli's Law

Bernoulli's law, named after the Swiss mathematician Daniel Bernoulli, is a principle of fluid dynamics that describes the behavior of a moving fluid. This fundamental theorem is crucial for understanding various phenomena in fluid mechanics, engineering, and even aviation. Bernoulli's law states that for an incompressible, frictionless fluid, the total mechanical energy remains constant along a streamline.

Bernoulli's law express energy conservation and work-energy theorem applied to the flow of the ideal fluid – incompressible and without friction.

Let us consider this ideal fluid flowing in a tube with variable cross-section area. Let us also assume that the ends of the tube are at different elevations (see Figure 1). The liquid flow is shown as the green arrow.

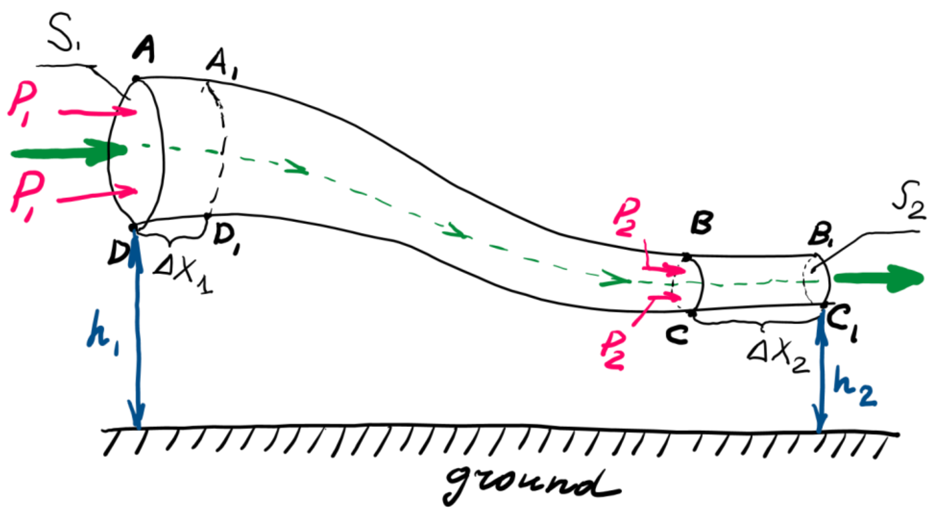

Figure 1.

Let us also assume that in a certain time period, the liquid in part ABCD shifts and occupies part $A_1B_1C_1D_1$. We will try to figure out how liquid velocity and pressure at the left end of the tube are related to these at the other end.

First, let us calculate the total work done by the pressure force on the liquid. At the left side, the pressure "pushes" the liquid to the right. The corresponding work A_1 is:

$$A_1 = P_1 \cdot S_1 \cdot \Delta x_1 = P_1 \cdot \Delta V_1 \qquad (1)$$

Here S1 is the tube cross-section area at the left side and DV_1 is the volume of cylinder AA_1D_1D. This volume can be represented as:

$$\Delta V_1 = \frac{\Delta m_1}{\varrho} \qquad (2),$$

where Δm_1 is the mass of the fluid in the cylinder AA_1D_1D, and ϱ is the fluid density. Substituting (2) into (1) we obtain:

$$A_1 = P_1 \cdot \frac{\Delta m_1}{\varrho} \qquad (3)$$

Similarly, we can calculate the work A_2 done by the liquid at the right side of the tube:

$$A_2 = P_2 \cdot \frac{\Delta m_2}{\varrho} \qquad (4)$$

Since the liquid is incompressible, $\Delta m_1 = \Delta m_2 = \Delta m$. The liquid density is also constant.

Total mechanical work W, done on the liquid in the tube ABCD is:

$$W = A_1 - A_2 = P_1 \cdot \frac{\Delta m}{\varrho} - P_2 \cdot \frac{\Delta m}{\varrho} \qquad (5)$$

This total work is equal to change of energy of the moved liquid. But the flow does not change in time (we call id steady-state flow), so the energy of liquid in part A_1BCD_1 did not change. The energy change is just the difference in energy ΔE of the liquid mass Δm in positions AA_1D_1D and BB_1C_1C :

$$\Delta E = (\epsilon_2 - \epsilon_1) \cdot \Delta m \qquad (6),$$

Where ϵ_2 and ϵ_1 are total energy of the fluid per unit mass at the left and right end of the tube. Equating (5) and (6) and dividing both parts of the equality by Δm we obtain:

$$\epsilon_2 - \epsilon_1 = \frac{P_1}{\varrho} - \frac{P_2}{\varrho} \quad \Rightarrow \quad \epsilon_1 + \frac{P_1}{\varrho} = \epsilon_2 + \frac{P_2}{\varrho} \qquad (7)$$

But is a sum of the kinetic and potential energy per unit mass:

$$\epsilon_1 = \frac{v_1^2}{2} + gh_1; \epsilon_2 = \frac{v_2^2}{2} + gh_2 \qquad (8)$$

Combining (7) and (8), we obtain:

$$\frac{v_1^2}{2} + gh_1 + \frac{P_1}{\varrho} = \frac{v_2^2}{2} + gh_2 + \frac{P_2}{\varrho} \qquad (9),$$

Where $v_{1,2}$ are the velocities of the fluid at the left and right end of the tube, and $h_{1.2}$ are elevations of the tube ends (Fig. 1).

Expression (9) is the mathematical formulation of the Bernoulli's law:

"For the steady -state flow of incompressible fluid

$$\frac{v^2}{2} + gh + \frac{P}{\varrho} = const \qquad (10)."$$

Bernoulli's law signifies that if a fluid's velocity increases, the pressure within the fluid decreases, and vice versa. This inverse relationship between fluid velocity and pressure is the essence of Bernoulli's law. Bernoulli's principle helps explain how airplanes generate lift. The air speed above the wing is faster than below it, creating a pressure difference that results in upward lift.

Problems:

1.Consider a pipe with varying diameters through which water flows steadily. At point A, the pipe has a diameter of 0.1 meters, and the water speed is 2 meters per second. The pressure at point A is 20000 Pa. At point B, the diameter of the pipe is 0.05 meters.

- Calculate the water speed at point B.
- Determine the pressure at point B, assuming the elevation remains constant.

2. An airplane wing has air flowing over it at different speeds. The speed of the airflow above the wing is 60 meters per second, while the speed below the wing is 40 meters per second. The pressure below the wing is 101325 Pa.

- Calculate the pressure above the wing using Bernoulli's law.

3. Water flows through a tube with a constant speed of 3 meters per second at point X, where the pressure is 25000 Pa and the elevation is 2 meters. At point Y, the elevation is 1 meter.

- Calculate the pressure at point Y, assuming the speed of the water remains constant.
- Explain how the change in elevation affects the pressure according to Bernoulli's law.