Homework for March 23, 2025 Math 8

Algebra.

Review the last algebra classwork handouts. Solve the unsolved problems from the previous homeworks. Try solving the following problems.

1. Assume that the set of rational numbers $\mathbb{Q}$ is divided into two subsets, $\mathbb{Q}_<$ and $\mathbb{Q}_>$, such that all elements of $\mathbb{Q}_>$ are larger than any element of $\mathbb{Q}_<$: $\forall a \in \mathbb{Q}_<, \forall b \in \mathbb{Q}_>, a < b$.
   a. Prove that if $\mathbb{Q}_>$ contains the smallest element, $\exists b_0 \in \mathbb{Q}_>, \forall b \in \mathbb{Q}_>, b_0 \leq b$, then $\mathbb{Q}_<$ does not contain the largest element
   b. Prove that if $\mathbb{Q}_<$ contains the largest element, $\exists a_0 \in \mathbb{Q}_<, \forall a \in \mathbb{Q}_<, a \leq a_0$, then $\mathbb{Q}_>$ does not contain the smallest element
   c. Present an example of such a partition, where neither $\mathbb{Q}_>$ contains the smallest element, nor $\mathbb{Q}_<$ contains the largest element
2. Prove the following properties of countable sets. For any two countable sets, A, B,
   a. Union, $A \cup B$, is also countable, $\big((c(A) = \aleph_0) \wedge (c(B) = \aleph_0)\big) \Rightarrow (c(A \cup B) = \aleph_0)$
   b. Product, $A \times B = \{(a, b), a \in A, b \in B\}$, is also countable, $\big((c(A) = \aleph_0) \wedge (c(B) = \aleph_0)\big) \Rightarrow (c(A \cup B) = \aleph_0)$
   c. For a collection of countable sets, $\{A_n\}, c(A_n) = \aleph_0$, the union is also countable, $c(A_1 \cup A_2 \dots \cup A_n) = \aleph_0$

3. Let W be the set of all "words" that can be written using the alphabet consisitng of 26 lowercase English letters; by a "word", we mean any (finite) sequence of letters, even if it makes no sense – for example, abababaaaaa. Prove that W is countable. [Hint: for any n, there are only finitely many words of length n.]
4. Compare the following real numbers (are they equal? which is larger?)
   a. 1.33333… = 1.(3) and 4/3
   b. 0.09999… = 0.0(9) and 1/10
   c. 99.9999… = 99.(9) and 100
   d. $\sqrt[2]{2}$ and $\sqrt[3]{3}$
5. Simplify the following real numbers. Are these numbers rational? (hint: you may use the formula for an infinite geometric series).

a. 1/1.1111…=1/1.1(1)
b. 2/1.2323…=2/1.23(23)
c. 3/0.123123…=3/0.123(123)

6. Write the following rational decimals in the binary system (hint: you may use the formula for an infinite geometric series).
   a. 1/8
   b. 2/7
   c. 0.1
   d. $0.33333\ldots = 0.(3)$
   e. $0.13333\ldots = 0.1(3)$
7. Try proving the following properties of real numbers and arithmetical operations on them using definition of a real number as the Dedekind section and the validity of these properties for rational numbers.

Ordering and comparison.

1. $\forall\ a, b \in \mathbb{R}$, one and only one of the following relations holds
   - $a = b$
   - $a < b$
   - $a > b$
2. $\forall\ a, b \in \mathbb{R}, \exists c \in \mathbb{R}, (c > a) \wedge (c < b)$, i.e. $a < c < b$
3. Transitivity. $\forall\ a, b, c \in \mathbb{R}, \{(a < b) \wedge (b < c)\} \Rightarrow (a < c)$
4. Archimedean property. $\forall\ a, b \in \mathbb{R}, a > b > 0, \exists n \in \mathbb{N}$, such that $a < nb$

Addition and subtraction.

- $\forall\ a, b \in \mathbb{R}, a + b = b + a$
- $\forall\ a, b, c \in \mathbb{R}, (a + b) + c = a + (b + c)$
- $\forall\ a \in \mathbb{R}, \exists 0 \in \mathbb{R}, a + 0 = a$
- $\forall\ a \in \mathbb{R}, \exists - a \in \mathbb{R}, a + (-a) = 0$
- $\forall\ a, b \in \mathbb{R}, a - b = a + (-b)$
- $\forall\ a, b, c \in \mathbb{R}, (a < b) \Rightarrow (a + c < b + c)$

Geometry.

Review the previous classwork notes. Solve the problems below and the remaining problems from the previous homework (some problems are repeated – skip the ones you have already done).

Problems.

1. Review derivation of the equation describing an ellipse and derive in a similar way,
    a. Equation of an ellipse, defined as the locus of points P for which the distance to a given point (focus F_2) is a constant fraction of the perpendicular distance to a given line, called the directrix, $|PF_2|/|PD| = e < 1$.
    b. Equation of a hyperbola, defined as the locus of points for which the ratio of the distances to one focus and to a line (called the directrix) is a constant e. However, for a hyperbola it is larger than 1, $|PF_2|/|PD| = e > 1$.
2. Find (describe) set of all points formed by the centers of the circles that are tangent to a given circle of radius r and a line at a distance $d > r$ from its center, O.
3. Using the method of coordinates, prove that the geometric locus of points from which the distances to two given points have a given ratio, $q \neq 1$, is a circle.
4. Find the equation of the locus of points equidistant from two lines, $y = ax + b$ and $y = mx + n$, where a, b, m, n are real numbers.
5. Find the distance between the nearest points of the circles,
    a. $(x-2)^2 + y^2 = 4$ and $x^2 + (y-1)^2 = 9$
    b. $(x+3)^2 + y^2 = 4$ and $x^2 + (y-4)^2 = 9$
    c. $(x-2)^2 + (y+1)^2 = 4$ and $(x+1)^2 + (y-3)^2 = 5$
    d. $(x-a)^2 + y^2 = r_1^2$ and $x^2 + (y-b)^2 = r_2^2$