January 26, 2025

Algebra.

Maps. Functions. Injections, surjections, bijections.

A **map** is a rule that associates unique objects to elements in a given set. A **function** is a map that uniquely associates to **every** element of one set some element of another set: $\forall a \in A, a \xrightarrow{f} f(a) = b \in B$. A **partial function** on set A maps a subset of elements from set A on elements from set B.

Definition. A function is a relation that uniquely associates every member a in set A with some member b in set B, i.e. a function f is a map $A \xrightarrow{f} B$ such that $\forall a \in A, \exists! \, b \in B, b = f(a)$.

Note that we do not require that every element $b \in B$ appears as a value of a function. A function therefore can be one-to-one or many-to-one relation.

Definition. The set A of values at which a function f is defined is called its **domain**, while the set $f(A)$ of values that the function can produce, which is a subset of B, $f(A) \subseteq B$, is called its **range**. The set B is called the **codomain** of f.

Definition. For a subset X of the domain A of function f, $X \subseteq A$, the **image**, $f(X)$, is the set of values $y \in B$, $y = f(x)$, $\forall x \in X$,

$$Y = f(X) = \{y: (y \in B) \land (\exists x \in X, y = f(x))\}$$

Definition. For a subset Y of the range B of function f, $Y \subseteq B$, the **pre-image**, $f^{-1}(Y)$, is the set of values $x \in A$, $y = f(x)$, $y \in Y$,

$$X = f^{-1}(Y) = \{x: (x \in A) \land (\exists y \in Y, f(x) = y)\}$$

In particular, if an image $Y = \{y\}$ is a single point $y \in B$, in this case, $f^{-1}(\{y\})$ is the set of all solutions of the equation, $f(x) = y$.

Exercise 1. Let function f map $A \xrightarrow{f} B$. Prove that for any two subsets of its domain, $X_1 \subset A, X_2 \subset A$, $f(X_1 \cup X_2) = f(X_1) \cup f(X_2)$.

Show that it could happen that $f(X_1 \cap X_2) \neq f(X_1) \cap f(X_2)$ (hint: take X_1, X_2 so that they do not intersect).

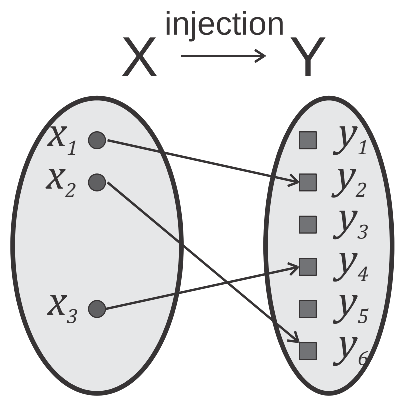
Exercise 2. Let function f map $A \xrightarrow{f} B$. Prove that for any two subsets of its co-domain, $Y_1 \subset B, Y_2 \subset B, f^{-1}(Y_1 \cap Y_2) = f^{-1}(Y_1) \cap f^{-1}(Y_2)$.

Definition. The function $A \xrightarrow{f} B$ is **injective** (one-to-one) if every element of the co-domain B is mapped to by at most one element of the domain A (has no more than one pre-image),

$$\forall (x_1, x_2) \in A, \big(f(x_1) = f(x_2)\big) \Rightarrow (x_1 = x_2), \text{ or,}$$

$$\forall (x_1, x_2) \in A, (x_1 \neq x_2) \Rightarrow \big(f(x_1) \neq f(x_2)\big)$$

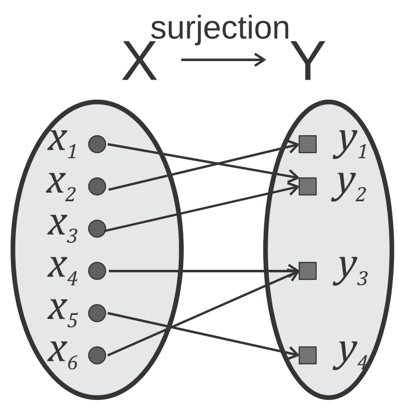
An injective function is an **injection**.



Definition. The function $A \xrightarrow{f} B$ is **surjective** (onto) if every element of the co-domain B is mapped to by at least one element of the domain A (has pre-image in A),

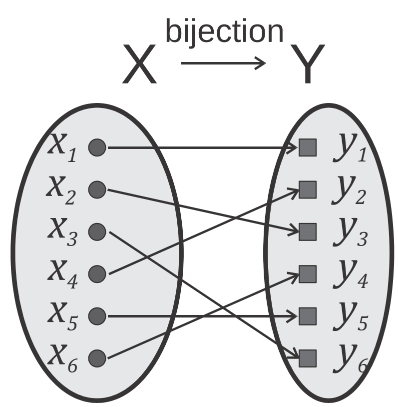
$$\forall y \in B, \exists x \in A: y = f(x).$$

That is, the image of the range of the surjective function coincides with the co-domain. A surjective function is a **surjection.**



An injective function need not be surjective (not all elements of the co-domain may have pre-images), and a surjective function need not be injective (some images may be associated with more than one pre-image).

Definition. The function $A \xrightarrow{f} B$ is **bijective** (one-to-one correspondence, or one-to-one and onto) if every element of the co-domain is mapped to by exactly one element of the domain. That is, the function is both injective and surjective. A bijective function is a **bijection.**



A function is bijective if and only if every possible image is mapped to by exactly one argument (pre-image),

$\forall y \in B, \exists!\, x \in A, y = f(x)$.

A function $A \xrightarrow{f} B$ is bijective if and only if it is invertible, that is, there exists a function g, $B \xrightarrow{g} A$ such that $\forall x \in A, g\big(f(x)\big) = x$, and $\forall y \in B, f\big(g(y)\big) = y$. Such a function is called inverse of f and denoted $g = f^{-1}$. This function maps each pre-image to its unique image. In other words, $g \circ f = g\big(f(x)\big)$ is an identity function on A, and $f \circ g = f\big(g(y)\big)$ is an identity function on B.

Bijections provide a way of comparing and identifying different sets. In particular, if there exists a bijection f between two finite sets A and B, then $|A| = |B|$.

Exercise 1. Show that f: $A \xrightarrow{f} B$ is not injective exactly when one can find $x_1, x_2 \in A$ such that $x_1 \neq x_2$, but $f(x_1) = f(x_2)$.

Exercise 2. Let $: A \xrightarrow{f} B$ and g : $B \xrightarrow{f} C$ be bijections. Prove that the composition $g \circ f : A \xrightarrow{g \circ f} C$, defined by g $\circ f(x) = g\big(f(x)\big)$, is also a bijection, and that so is $(g \circ f)^{-1} = (f)^{-1} \circ (g)^{-1}$.

Exercise 3. Construct bijections between the following sets:

1. (subsets of the set {1, . . . , n}) ↔ (sequences of zeros and ones of length n)
2. (5-element subsets of {1, . . . , 15})↔ (10-element subsets of {1, . . . , 15})
3. [set of all ways to put 10 books on two shelves (order on each shelf matters)]↔ (set of all ways of writing numbers 1, 2, . . . , 11 in some order) [Hint: use numbers 1. . . 10 for books and 11 to indicate where one shelf ends and the other begins.]
4. (all integer numbers)↔ (all even integer numbers)
5. (all positive integer numbers)↔ (all integer numbers)
6. (interval (0,1))↔ (interval (0,5))
7. (interval (0,1))↔ (halfline $(1, \infty)$) [Hint: try $1/x$.]
8. (interval (0,1))↔ (halfline $(0, \infty)$)

9. (all positive integer numbers)$\leftrightarrow$ (all integer numbers)

Exercise 4. Let A be a finite set, with 10 elements. How many bijections $f : A \to A$ are there? What if A has n elements?

Exercise 5. Let f: $\mathbb{Z} \to \mathbb{Z}$ be given by $f(n) = 2n$. Is this function injective? surjective?