Homework for January 12, 2025.

Algebra.

Review the last classwork handout. Review and solve the classwork exercises which were not solved and unsolved problems from the previous homeworks. Solve the following problems (skip the ones that you have already solved).

1. Prove the following properties of the Cartesian product,
   a. $A \times (B \cap C) = (A \times B) \cap (A \times C)$
   b. $A \times (B \cup C) = (A \times B) \cup (A \times C)$
   c. $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$
2. Find the Cartesian product, $A \times B$, of the following sets,
   a. $A = \{a, b\}$, $B = \{\uparrow, \downarrow\}$
   b. $A = \{June, July, August\}$, $B = \{1,15\}$
   c. $A = \emptyset$, $B = \{1,2,3,4,5,6,7,8,9\}$
3. Describe the set of points determined by the Cartesian product, $A \times B$, of the following sets (illustrate schematically on a graph),
   a. $A = [0,1]$, $B = [0,1]$ (two segments from 0 to 1)
   b. $A = [-1,1]$, $B = (-\infty, \infty)$
   c. $A = (-\infty, 0]$, $B = [0, \infty)$
   d. $A = (-\infty, \infty)$, $B = (-\infty, \infty)$
   e. $A = [0,1)$, $B = \mathbb{Z}$ (set of all integers)
4. Propose 3 meaningful examples of a Cartesian product of two sets.
5. $n_A = |A|$ is the number of elements in a set A.
   a. What is the number of elements in a set $A \times A$
   b. What is the number of elements in a set $A \times (A \times A)$

<u>Bonus: recap problems from previous homeworks – solve the ones you have not yet solved</u>

6. Using the method of mathematical induction, prove the following equality,

$$\sum_{k=0}^{n} k \cdot k! = (n+1)! - 1$$

7. Put the sign $<$, $>$, or $=$, in place of … below,

$$\frac{n+1}{2} \ldots \sqrt[n]{n!}$$

8. Find the following sum.

$$\left(2+\frac{1}{2}\right)^2+\left(4+\frac{1}{4}\right)^2+\cdots+\left(2^n+\frac{1}{2^n}\right)^2$$

9. The lengths of the sides of a triangle are three consecutive terms of the geometric series. Is the common ratio of this series, q, larger or smaller than 2?

10. Solve the following equation,

$$\frac{x-1}{x}+\frac{x-2}{x}+\frac{x-3}{x}+\cdots+\frac{1}{x}=3, \text{ where } x \text{ is a positive integer.}$$

11. Find the following sum,
   a. $1+2\cdot 3+3\cdot 7+\cdots+n\cdot(2^n-1)$
   b. $1\cdot 3+3\cdot 9+5\cdot 27+\cdots+(2n-1)\cdot 3^n$

12. Numbers $a_1, a_2, \ldots, a_n$ are the consecutive terms of a geometric progression, and the sum of its first n terms is S_n. Show that,

$$S_n = a_1 a_n\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right)$$

13. Prove that three terms shown below are the three terms of the geometric progression, and find the sum of its first n terms, beginning with the first one below,

$$\frac{\sqrt{3}+1}{\sqrt{3}-1}+\frac{1}{3-\sqrt{3}}+\frac{1}{6}+\cdots$$

14. What is the maximum value of the expression, $(1+x)^{36}+(1-x)^{36}$ in the interval $|x| \le 1$?

15. Find the coefficient multiplying x^9 after all parentheses are expanded in the expression, $(1+x)^9+(1+x)^{10}+\cdots+(1+x)^{19}$.

Geometry.

Review the previous classwork notes on the method of coordinates. No new geometry problems: please try solving the unsolved problems from the last homework, which are repeated below.

Problems.

1. Review the solution of the radical axis of two circles problem: find the locus of points whose powers with respect to two non-concentric circles are equal. Consider situation when circles are concentric.
2. Complete the following exercises from class. Find the locus of points satisfying each of the following equations or inequalities (graph it on a coordinate plane).
   a. $|x| = |y|$
   b. $|x| + x = |y| + y$
   c. $|x|/x = |y|/y$
   d. $[y] = [x]$
   e. $\{y\} = \{x\}$
   f. $x^2 - y^2 \geq 0$
   g. $x^2 + y^2 \leq 1$
   h. $x^2 + 8x = 9 - y^2$
3. Describe the locus of all points (x, y) equidistant to the X-axis (i. e. the line $y = 0$) and a given point P $(0,2)$ on the Y-axis. Write the formula relating y and x for these points.
4. (Skanavi 15.105) Find the (x, y) coordinates of the vertex C of an equilateral triangle ABC if A and B have coordinates $A(1,3)$ and $B(3,1)$, respectively.
5. (Skanavi 15.106) Find the (x, y) coordinates of the vertices C and D of a square $ABCD$ if A and B have coordinates $A(2,1)$ and $B(4,0)$, respectively.

6. *Prove that the length of the bisector segment BB' of the angle $\angle B$ of a triangle ABC satisfies $|BB'|^2 = |AB||BC| - |AB'||B'C|$.
7. **Prove the following Ptolemy's inequality. Given a quadrilateral $ABCD$,

$$|AC| \cdot |BD| \leq |AB| \cdot |CD| + |BC| \cdot |AD|$$

Where the equality occurs if $ABCD$ is inscribable in a circle.