Homework for October 20, 2024 Math 9

Algebra.

Review the classwork handout and complete the exercises. Solve the remaining problems from the previous homework. Solve the following problems (you may skip the ones considered in class).

1. Prove that the following number is irrational.
   a. $\sqrt{3}$
   b. $\sqrt{5}$
2. Prove that for $m, n \geq 0$,
   a. $\frac{m+n}{2} \geq \sqrt{mn}$
   b. $mn(m+n) \leq m^3 + n^3$
3. There are four consecutive members of a geometric progression. Sum of the first and the fourth is 13, sum of the second and the third is 4. Find these four numbers.
4. Consider the progression

   1, 3, 5, 7, …, 993, 995, 997, 999.

   How many terms does it have? Find the sum of these terms.

5. Prove that the sum of the n first odd numbers is a perfect square ($1=1^2$, $1+3=2^2$, $1+3+5=3^2$ etc.).
6. Find the sum of the first n even numbers and prove the result using mathematical induction.

Recap. In order to prove the equality $A(n) = B(n)$, for any n, using the method of mathematical induction you have to

a. Prove that $A(1) = B(1)$
b. Prove that $A(k+1) - A(k) = B(k+1) - B(k)$ (*)
c. Then from assumption $A(k) = B(k)$ and from equality (*) follows $A(k+1) = B(k+1)$

7. Prove by mathematical induction that
   a. for any natural number n, $15^n + 6$ is divisible by 7.
   b. $\forall n, \exists k, 5^n + 3 = 4k$

c. $\forall x > -1, \forall n \geq 2, \quad (1+x)^n \geq 1 + nx$

d. $\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

e. $\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

f. $\sum_{k=1}^{n} \frac{1}{k^2+k} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n\cdot(n+1)} = \frac{n}{n+1}$

g. $\sum_{k=2}^{n} \frac{1}{k^2-1} = \frac{1}{1\cdot 3} + \frac{1}{2\cdot 4} + \frac{1}{3\cdot 5} + \cdots + \frac{1}{(n-1)\cdot(n+1)} = \frac{3}{4} - \frac{2n+1}{2n(n+1)}$

8. **Recap**. There are 15 students in a class.
   a. Each student needs to make a presentation of a problem. How many ways are there to arrange the order in which they make presentations – i. e. decide who speaks first, who speaks second, third, …, fifteenth?
   b. How many ways is there to select a pair of students of whom one will present solution of an algebra problem, and the other of a problem in geometry?
   c. How many ways is there to select a team of two students who will represent the class at a math competition?

Geometry.

Review the classwork handout. Review constructing the segments equal to the ratio and the product of two given segments. Solve the following problems using the Thales theorem which had been proven in class. The problems are from the previous assignment – skip those which you have already solved.

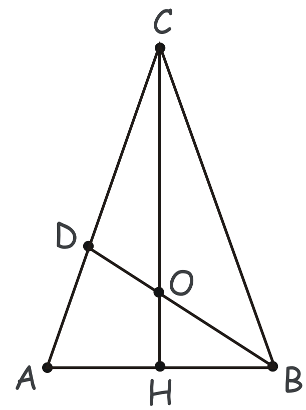
Problems.

1. Given two segments of <u>integer</u> lengths a and b, construct,
   a. A segment of length a^2
   b. A segment of length $1/a$
   c. A segment of length ab
   d. A segment of length a/b

   Hint: construct these segments given a, b, and a segment of length 1.

2. Prove that medians of a triangle divide one another in the ratio 2:1, in other words, the medians of a triangle "trisect" one another (Coxeter, Gretzer, p.8).

3. In isosceles triangle ABC point D divides the side AC into segments such that |AD|:|CD|=1:2. If CH is the altitude of the triangle and point O is the intersection of CH and BD, find the ratio |OH| to |CH|.



4. In a trapezoid $ABCD$ with the bases $|AB| = a$ and $|CD| = b$, segment MN parallel to the bases, $MN||AB$, connects the opposing sides, $M \in [AD]$ and $N \in [BC]$. MN also passes through the intersection point O of the diagonals, AC and BD, as shown in the Figure. Prove that $|MN| = \frac{2ab}{a+b}$.