

Geometry. Trigonometry.

Trigonometric formulas and equations.

Using the formulas for the sine and cosine of the sum/difference of two angles, which we have previously derived,

$$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$$

$$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$$

It is easy to obtain all other trigonometric formulae.

Exercise. Derive the following expressions for the products of sine and cosine,

$$\sin \alpha \sin \beta = \frac{1}{2} [\cos(\alpha - \beta) - \cos(\alpha + \beta)]$$

$$\cos \alpha \cos \beta = \frac{1}{2} [\cos(\alpha - \beta) + \cos(\alpha + \beta)]$$

$$\sin \alpha \cos \beta = \frac{1}{2} [\sin(\alpha - \beta) + \sin(\alpha + \beta)]$$

Solution. These expressions are obtained by adding and subtracting the above expressions for $\sin(\alpha \pm \beta)$, $\cos(\alpha \pm \beta)$. For example,

$$\sin(\alpha + \beta) + \sin(\alpha - \beta) = 2 \sin \alpha \cos \beta, \text{ etc.}$$

Exercise. Derive the following expressions for sums and differences of sine and cosine,

$$\sin \alpha + \sin \beta = 2 \sin \left[\frac{1}{2} (\alpha + \beta) \right] \cos \left[\frac{1}{2} (\alpha - \beta) \right]$$

$$\sin \alpha - \sin \beta = 2 \cos \left[\frac{1}{2} (\alpha + \beta) \right] \sin \left[\frac{1}{2} (\alpha - \beta) \right]$$

$$\cos \alpha + \cos \beta = 2 \cos \left[\frac{1}{2} (\alpha + \beta) \right] \cos \left[\frac{1}{2} (\alpha - \beta) \right]$$

$$\cos \alpha - \cos \beta = -2 \sin \left[\frac{1}{2}(\alpha + \beta) \right] \sin \left[\frac{1}{2}(\alpha - \beta) \right]$$

Solution. The above expressions are obtained by representing α and β as, $\alpha = \frac{1}{2}(\alpha + \beta) + \frac{1}{2}(\alpha - \beta)$, $\alpha = \frac{1}{2}(\alpha + \beta) - \frac{1}{2}(\alpha - \beta)$, and using the previously obtained expressions for $\sin(\alpha \pm \beta)$, $\cos(\alpha \pm \beta)$.

Exercise. Derive the following expressions,

$$\tan \alpha \pm \tan \beta = \frac{\sin(\alpha \pm \beta)}{\cos(\alpha) \cos(\beta)} \quad \cot \alpha \pm \cot \beta = \pm \frac{\sin(\alpha \pm \beta)}{\sin(\alpha) \sin(\beta)}$$

$$\sin^2 \alpha = \frac{1}{2}(1 - \cos 2\alpha) \quad \cos^2 \alpha = \frac{1}{2}(1 + \cos 2\alpha)$$

$$\sin 3\alpha = 3 \sin \alpha - 4 \sin^3 \alpha \quad \cos 3\alpha = 4 \cos^3 \alpha - 3 \cos \alpha$$

$$\sin^3 \alpha = \frac{1}{4}(3 \sin \alpha - \sin 3\alpha) \quad \cos^3 \alpha = \frac{1}{4}(3 \cos \alpha + \cos 3\alpha)$$

$$\sin \frac{\alpha}{2} = \sqrt{\frac{1}{2}(1 - \cos \alpha)} \quad \cos \frac{\alpha}{2} = \sqrt{\frac{1}{2}(1 + \cos \alpha)}$$

$$\tan \frac{\alpha}{2} = \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} = \frac{1 - \cos \alpha}{\sin \alpha} = \frac{\sin \alpha}{1 + \cos \alpha}$$

$$\cot \frac{\alpha}{2} = \sqrt{\frac{1 + \cos \alpha}{1 - \cos \alpha}} = \frac{1 + \cos \alpha}{\sin \alpha} = \frac{\sin \alpha}{1 - \cos \alpha}$$

$$\sin 2\alpha = \frac{2 \tan \alpha}{1 + \tan^2 \alpha}$$

$$\cos 2\alpha = \frac{1 - \tan^2 \alpha}{1 + \tan^2 \alpha}$$

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$$

Trigonometric functions and relations.

Exercise. Fill in the table of the trigonometric functions of complementary angles below.

α	$\sin \alpha$	$\cos \alpha$	$\tan \alpha$	$\cot \alpha$
$\frac{\pi}{2} - \alpha$	$\sin\left(\frac{\pi}{2} - \alpha\right) = \cos \alpha$			
$\frac{\pi}{2} + \alpha$				
$\pi - \alpha$				
$\pi + \alpha$				
$\frac{3}{2}\pi - \alpha$				
$\frac{3}{2}\pi + \alpha$				
$-\alpha$				



