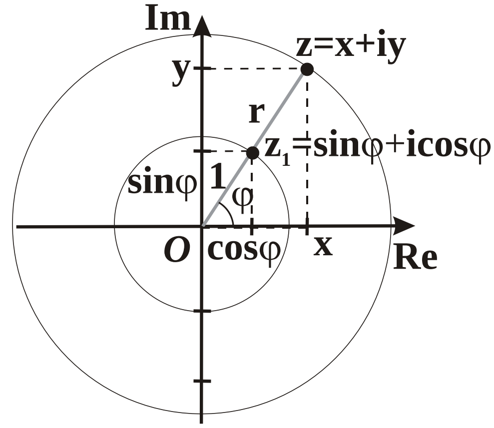
May 11, 2025 Math 9

Algebra.

Trigonometric form of complex numbers. Geometric interpretation.



Let us consider complex numbers with the absolute value of 1,

$z_1 = x_1 + iy_1$, $|z_1|^2 = z_1\overline{z_1} = {x_1}^2 + {y_1}^2 = 1$.

There is an obvious one-to-one correspondence between such numbers and points $Z_1(x_1, y_1)$ on a circle of unit radius. Hence, we can express such numbers in terms of an angle, φ, parameterizing points on the unit circle,

$z_1 = x_1 + iy_1 = \cos\varphi + i\sin\varphi$.

More generally, any complex number, $z = x + iy$, whose absolute value is $|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2} = r$, can be written in the trigonometric form as,

$z = x + iy = r(\cos\varphi + i\sin\varphi)$.

Geometrically, it is represented by a point $Z(x, y)$ on a circle of radius $r = |z|$. Position of this point is specified by an angle, φ, which is conventionally measured counterclockwise from the positive direction of the X-axis. Angle φ is called the argument of the complex number z and is denoted $\varphi = Arg(z)$. Thus, instead of describing a complex number by its real and imaginary part, i.e. its coordinates, (x, y), we can describe it by its magnitude and argument (polar coordinates), (r, φ), where $r \geq 0$ and $0 \leq \varphi = Arg(z) < 360°$.

It is now easy to prove the following important property of the multiplication of complex numbers.

Theorem. When we multiply two complex numbers, magnitudes multiply and arguments add,

$|z_1 z_2| = |z_1||z_2|$, $Arg(z_1 z_2) = \big(Arg(z_1) + Arg(z_2)\big) mod\ 360°$.

Proof. Let $Arg(z_1) = \varphi_1$ and $Arg(z_2) = \varphi_2$, so $z_1 = |z_1|(\cos\varphi_1 + i\sin\varphi_1)$ and $z_2 = |z_2|(\cos\varphi_2 + i\sin\varphi_2)$. Perform the multiplication directly,

$$z_1 z_2 = |z_1|(\cos\varphi_1 + i\sin\varphi_1)|z_2|(\cos\varphi_2 + i\sin\varphi_2) =$$

$$|z_1||z_2|\big(\cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2 + i(\sin\varphi_1\cos\varphi_2 + \cos\varphi_1\sin\varphi_2)\big)$$
$$= |z_1||z_2|(\cos(\varphi_1+\varphi_2) + i\sin(\varphi_1+\varphi_2))$$

Complex numbers whose arguments would differ by multiples of 360° are identical and correspond to the same point on the complex plane. Hence, the argument is computed $mod\ 360°$, ensuring that $0 \le \varphi_1 + \varphi_2 < 360°$.

Theorem. Multiplication of a complex number, $z = x + iy = r(\cos\varphi + i\sin\varphi)$, by a complex number of unit magnitude and argument ψ,

$z_\psi = x_\psi + iy_\psi = \cos\psi + i\sin\psi$,

corresponds to a counterclockwise rotation of the point, $Z(x, y)$, on the complex plane, by an angle ψ,

$|zz_\psi| = r,\ Arg(zz_\psi) = \varphi + \psi$.

Proof. Indeed, perform the multiplication directly,

$$zz_\psi = (x+iy)(x_\psi + iy_\psi) = r(\cos\varphi + i\sin\varphi)(\cos\psi + i\sin\psi)$$
$$= r\big(\cos\varphi\cos\psi - \sin\varphi\sin\psi + i(\sin\varphi\cos\psi + \cos\varphi\sin\psi)\big)$$
$$= r(\cos(\varphi+\psi) + i\sin(\varphi+\psi))$$

It is clear that multiplication by a complex number with magnitude r' and argument ψ is equivalent to the combination of multiplication by a number of unit magnitude and argument ψ, and y the real number r'.

Theorem. Multiplication of a complex number, $z = x + iy$, by a complex number with magnitude r' and argument ψ,

$w = r'(\cos\psi + i\sin\psi)$,

results in a point on the complex plane, which is obtained from the point $Z(x, y)$ by the combination of a rotation by angle ψ and a homothety (rescaling) with scale factor r'.

Multiplication of all complex numbers by a complex number $w = r'(\cos\psi + i\sin\psi)$ is a transformation of the complex plane, which maps complex plane on itself. Identifying multiplication by a complex number with such transformation, we can state the following.

Theorem. Multiplication by a complex number with magnitude r' and argument ψ, $w = r'(\cos\psi + i\sin\psi)$, is a combination of rotation by angle ψ and homothety (rescaling) with scale factor r'.

De Moivre's formula.

Theorem. The formula named after Abraham de Moivre states that for any complex number, $z = x + iy = r(\cos\varphi + i\sin\varphi)$, and for any integer $n \in \mathbb{N}$,

$$z^n = \left(r(\cos\varphi + i\sin\varphi)\right)^n = r^n(\cos n\varphi + i\sin n\varphi)$$

Proof 1 (Mathematical induction).

1. Base case, $n = 1$: $z^1 = r(\cos\varphi + i\sin\varphi)$ is true.
2. $I(n) => I(n+1)$. Assume $z^n = r^n(\cos n\varphi + i\sin n\varphi)$ is true. Then,

$$\begin{aligned} z^{n+1} = z \cdot z^n &= r(\cos\varphi + i\sin\varphi) \cdot \left(r(\cos\varphi + i\sin\varphi)\right)^n \\ &= r(\cos\varphi + i\sin\varphi)r^n(\cos n\varphi + i\sin n\varphi) \\ &= r^{n+1}\big(\cos\varphi\cos n\varphi - \sin\varphi\sin n\varphi \\ &+ i(\sin\varphi\cos n\varphi + \cos\varphi\sin n\varphi)\big) \\ &= r^{n+1}(\cos(n+1)\varphi + i\sin(n+1)\varphi) \end{aligned}$$

Proof 2 (Geometrical).

$z^n = \left(r(\cos\varphi + i\sin\varphi)\right)^n = r(\cos\varphi + i\sin\varphi)r(\cos\varphi + i\sin\varphi) \dots . r(\cos\varphi + i\sin\varphi)$

By property of the multiplication of complex numbers, absolute values multiply, while arguments add. Therefore,

$|z^n| = |z|^n = r^n$, and $Arg(z^n) = nArg(z) = n\varphi$, wherefrom it follows that $z^n = r^n(\cos n\varphi + i\sin n\varphi)$.

n-th root.

The formula of de Moivre allows us to compute n -th root of a complex number. Suppose we want to solve the equation,

$$w^n = z$$

where $w, z \in \mathbb{C}$, so w is the n-th root of z. According to de Moivre's formula, if $w = |w|(\cos\psi + i\sin\psi)$, then $w^n = |w|^n(\cos n\psi + i\sin n\psi)$. Denoting $z = r(\cos\varphi + i\sin\varphi)$, we can rewrite the equation as,

$$w^n = |w|^n(\cos n\psi + i\sin n\psi) = r(\cos\varphi + i\sin\varphi)$$

One obvious solution is $r = |w|^n$ and $\varphi = n\psi$, $w = \sqrt[n]{r}\left(\cos\frac{\varphi}{n} + i\sin\frac{\varphi}{n}\right)$. However, because $\varphi = Arg(z) = Arg(w^n)$ and $\psi = Arg(w)$ are determined modulo 360° (2π radians), there are other solutions, too, satisfying the above equation, such as $w = \sqrt[n]{r}\left(\cos\frac{\varphi+2\pi}{n} + i\sin\frac{\varphi+2\pi}{n}\right)$. Generally, we must have $r = |w|^n$, and $\varphi = Arg(z) = Arg(w^n)\ mod\ 360° = nArg(w)\ mod\ 360° = n\psi\ mod\ 360°$. Altogether, there are n solutions,

$$w = \sqrt[n]{r}\left(\cos\frac{\varphi + 2\pi k}{n} + i\sin\frac{\varphi + 2\pi k}{n}\right), 0 \leq k < n$$

This is a special case of the following extremely important result, called the fundamental theorem of algebra.

Theorem. Any polynomial with complex coefficients of degree n has exactly n roots (counting with multiplicities).

There is no simple proof of this theorem (and, in fact, no purely algebraic proof: all the known proofs use some geometric arguments).

In particular, since any polynomial with real coefficients can be considered as a special case of a polynomial with complex coefficients, this shows that any real polynomial of degree n has exactly n complex roots.