April 6, 2025 Math 9

Algebra.

Polynomials and factorization.

Polynomial is an expression containing variables denoted by some letters, and combined using addition, multiplication and numbers. General form of the n-th degree polynomial of one variable x is,

$$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2 x^2 + a_1 x^1 + a_0. \qquad (1)$$

This includes quadratic polynomial for $n = 2$, cubic for $n = 3$, etc. The general form for the case of more than one variable is quite complex. For example,

$$P_n(x, y) = a_{n,0}x^n + a_{n-1,0}x^{n-1} + \cdots + a_{1,0}x^1 + a_{0,0} + a_{n-1,1}x^{n-1}y + \cdots + a_{1,1}xy + a_{0,1}y + \cdots$$

One should distinguish variables, such as x and y, which can take any real values, and the coefficients denoted here by a_n, etc, which are just fixed numbers defining a particular polynomial.

We consider only polynomials with one variable. The number, n, which is the highest power of x appearing (with non-zero coefficient) in the expression of a polynomial P, is called degree of P and often denoted deg (P).

One can add, subtract, and multiply polynomials in the obvious way. It is easy to see that for a product of two polynomials, P and Q,

$$\deg(PQ) = \deg(P) + \deg(Q)$$

However, in general one cannot divide polynomials: expression $\frac{x^3+3}{x^2+x-1}$ is not a polynomial. However, much like with integers, there is "division with remainder" for polynomials, also known as "long division".

Polynomial division transformation

Theorem. Let $D(x)$ be a polynomial with deg $(D) > 0$ (i.e., D is not a constant). Then any polynomial $P(x)$ can be uniquely written in the form

$$P(x) = D(x)Q(x) + R(x)$$

where $Q(x)$, $R(x)$ are polynomials, and $\deg(R) < \deg(D)$. The polynomial $R(x)$ is called the remainder upon division of $P(x)$ by $D(x)$.

Polynomial division allows for a polynomial to be written in a divisor–quotient form, which is often advantageous. Consider polynomials $P(x)$, $D(x)$ where $\deg(D) < \deg(P)$. Then, for some quotient polynomial $Q(x)$ and remainder polynomial $R(x)$ with $\deg(R) < \deg(D)$,

$$\frac{P(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)} \Leftrightarrow P(x) = D(x)Q(x) + R(x)$$

This rearrangement is known as the **division transformation** and derives from the corresponding arithmetical identity.

Polynomial long division algorithm for dividing a polynomial by another polynomial of the same or lower degree, is a generalized version of the familiar arithmetic technique called long division. It can be done easily by hand, because it separates an otherwise complex division problem into smaller ones.

Example

Find $\dfrac{x^3 - 12x^2 - 42}{x - 3}$.

The problem is written like this:

$$\frac{x^3 - 12x^2 + 0x - 42}{x - 3}.$$

The quotient and remainder can then be determined as follows:

1. Divide the first term of the numerator by the highest term of the denominator (meaning the one with the highest power of x, which in this case is x). Place the result above the bar ($x^3 \div x = x^2$).

$$\begin{array}{r} x^2 \\ x - 3 \overline{) x^3 - 12x^2 + 0x - 42} \end{array}$$

2. Multiply the denominator by the result just obtained (the first term of the eventual quotient). Write the result under the first two terms of the numerator ($x^2 \cdot (x - 3) = x^3 - 3x^2$).

$$\begin{array}{r l}
 & x^2 \\ \cline{2-2}
x-3 \big) & x^3 - 12x^2 + 0x - 42 \\
 & x^3 - 3x^2
\end{array}$$

3. Subtract the product just obtained from the appropriate terms of the original numerator (being careful that subtracting something having a minus sign is equivalent to adding something having a plus sign), and write the result underneath ($(x^3 - 12x^2) - (x^3 - 3x^2) = -12x^2 + 3x^2 = -9x^2$) Then, "bring down" the next term from the numerator.

$$\begin{array}{r l}
 & x^2 \\ \cline{2-2}
x-3 \big) & x^3 - 12x^2 + 0x - 42 \\
 & \underline{x^3 - 3x^2} \\
 & \qquad -9x^2 + 0x
\end{array}$$

4. Repeat the previous three steps, except this time use the two terms that have just been written as the numerator.

$$\begin{array}{r l}
 & x^2 - 9x \\ \cline{2-2}
x-3 \big) & x^3 - 12x^2 + 0x - 42 \\
 & \underline{x^3 - \ \ 3x^2} \\
 & \qquad -9x^2 + 0x \\
 & \qquad \underline{-9x^2 + 27x} \\
 & \qquad\qquad\quad -27x - 42
\end{array}$$

5. Repeat step 4. This time, there is nothing to "pull down".

$$
\begin{array}{r}
x^2 - 9x - 27 \\
x-3\overline{\big)\, x^3 - 12x^2 + 0x - 42} \\
\underline{x^3 - 3x^2} \qquad\qquad \\
-9x^2 + 0x \qquad \\
\underline{-9x^2 + 27x} \qquad \\
-27x - 42 \\
\underline{-27x + 81} \\
-123
\end{array}
$$

6. The polynomial above the bar is the quotient, and the number left over (-123) is the remainder.

$$\frac{x^3 - 12x^2 - 42}{x-3} = \underbrace{x^2 - 9x - 27}_{q(x)} \underbrace{-\frac{123}{x-3}}_{r(x)/g(x)}$$

The long division algorithm for arithmetic can be viewed as a special case of the above algorithm, in which the variable x is replaced by the specific number 10.

Little Bézout's (polynomial remainder) theorem. Factoring polynomials.

Theorem. The remainder of a polynomial $P(x)$ divided by a linear divisor $(x - a)$ is equal to $P(a)$.

The polynomial remainder theorem follows from the definition of polynomial long division; denoting the divisor, quotient and remainder by, respectively, $G(x)$, $Q(x)$, and $R(x)$, polynomial long division gives a solution of the equation

$$P(x) = Q(x)G(x) + R(x)$$

where the degree of $R(x)$ is less than that of $G(x)$. If we take $G(x) = x - a$ as the divisor, giving the degree of $R(x)$ as 0, i.e. $R(x) = r$,

$$P(x) = Q(x)(x - a) + r. \qquad (2)$$

Here r is a number. Setting $x = a$, we obtain $P(a) = r$.

Roots of polynomials.

Definition 1. A number $a \in \mathbb{R}$ is called a **root** of polynomial $P(x)$ if $P(a) = 0$.

Definition 2. A number $a \in \mathbb{R}$ is called a **multiple root** of polynomial $P(x)$ of multiplicity m if $P(x)$ is divisible (without remainder) by $(x-a)^m$ and not divisible by $(x-a)^{m+1}$.

If x_1 is the root of a polynomial $P_n(x)$ of degree n, then $r = 0$, and

$$P_n(x) = (x - x_1)Q_{n-1}(x), \tag{3}$$

where $Q_{n-1}(x)$ is a polynomial of degree n – 1**.** $Q_{n-1}(x)$ is simply the quotient, which can be obtained using the **polynomial long division**. Since x_1 is known to be the root of $P_n(x)$, it follows that the remainder r must be zero.

If we know m roots, $\{x_1, x_2, \dots x_m\}$, of a polynomial $P_n(x)$ (why is it obvious that $m \le n$?), then, applying the above reasoning recursively,

$$P_n(x) = (x - x_1)(x - x_2) \dots (x - x_m)Q_{n-m}(x), \tag{4}$$

So, if we know that $P_n(x)$ given by (1) has n roots, $\{x_1, x_2, \dots x_n\}$, then,

$$P_n(x) = a_n(x - x_1)(x - x_2) \dots (x - x_n). \tag{5}$$

If two polynomials,

$$P_n(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_2x^2 + a_1x^1 + a_0$$

and

$$Q_n(x) = b_n x^n + b_{n-1}x^{n-1} + b_{n-2}x^{n-2} + \dots + b_2x^2 + b_1x^1 + b_0$$

are equal, $P_n(x) = Q_n(x)$, then all corresponding coefficients are equal,

$$a_n = b_n, a_{n-1} = b_{n-1}, a_{n-2} = b_{n-2}, \dots, a_{n-m} = b_{n-m}, \dots, a_1 = b_1, a_0 = b_0. \tag{6}$$

This is the easiest way to obtain Vieta's theorem and its generalizations for higher-order polynomials.

Vieta theorem.

Theorem. Let $f(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x^0 + a_0$ be a polynomial with leading coefficient 1 and roots $x_1, x_2, \dots, x_n$,

$f(x) = (x - x_1)(x - x_2) \dots (x - x_n)$.

Then the coefficients of $f(x)$ can be written in terms of roots,

$$a_0 = (-1)^n x_1 x_2 \dots x_n$$

$$a_1 = (-1)^{n-1}(x_1 x_2 \dots x_{n-1} + x_1 x_2 \dots x_{n-2} x_n + \cdots + x_2 x_3 \dots x_n)$$

$$\dots$$

$$a_{n-1} = -(x_1 + x_2 + \cdots + x_n)$$

For $n = 2$, quadratic equation, $x^2 + px + q = (x - x_1)(x - x_2)$, we have,

$q = x_1 x_2$ and $p = -(x_1 + x_2)$

For the cubic equation, $n = 3$, where x_1, x_2 and x_3 are the roots,

$x^3 + a_2 x^2 + a_1 x + a_0 = (x - x_1)(x - x_2)(x - x_3)$,

$a_0 = -x_1 x_2 x_3$, $a_1 = x_1 x_2 + x_2 x_3 + x_1 x_3$, $a_2 = -(x_1 + x_2 + x_3)$

Moreover, any expression in the roots $x_1, x_2, \dots, x_n$ which is symmetric (i.e., doesn't change when we permute any two roots) can be written in terms of the coefficients $a_0, a_1, \dots, a_n$. Example: for $n = 2$, $x_1^2 + x_2^2 = \cdots$