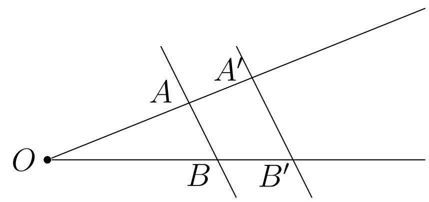
MATH 8B: HANDOUT 18 [MAR 2, 2024]
EUCLIDEAN GEOMETRY 8: SIMILAR TRIANGLES. THALES'S THEOREM.

Thales's Theorem



Theorem 32 (Thales's Theorem)**.** *Let points A', B' be on the sides of angle $\angle AOB$ as shown in the picture. Then lines AB and $A'B'$ are parallel if and only if*

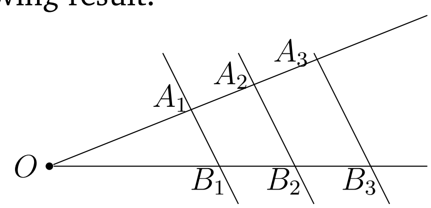
$$\frac{OA}{OB} = \frac{OA'}{OB'}$$

In this case, we also have $\frac{OA}{OB} = \frac{AA'}{BB'}$

We have already seen and proved a special case of this theorem when discussing the midline of a triangle.

The proof of this theorem is unexpectedly hard. In the case when $\frac{OA}{OA'}$ is a rational number, one can use arguments similar to those we did when talking about midline. The case of irrational numbers is harder yet. We skip the proof for now; it will be discussed in Math 9.

As an immediate corollary of this theorem, we get the following result.



Theorem 33. *Let points $A_1, \dots, A_n$ and $B_1, \dots B_n$ on the sides of an angle be chosen so that $A_1A_2 = A_2A_3 = \dots = A_{n-1}A_n$, and lines A_1B_1, A_2B_2, ... are parallel. Then $B_1B_2 = B_2B_3 = \dots = B_{n-1}B_n$.*

The proof of this theorem is left to you as an exercise.

Similar triangles

Definition. Two triangles $\triangle ABC$, $\triangle A'B'C'$ are called *similar* (denoted $\triangle ABC \sim \triangle A'B'C'$) if

$$\angle A \cong \angle A', \quad \angle B \cong \angle B', \quad \angle C \cong \angle C'$$

and the corresponding sides are proportional, i.e.

$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}$$

The common ratio $\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}$ is sometimes called the similarity coefficient.

There are some similarity tests:

Theorem 34 (AA(A) similarity test)**.** *If the corresponding angles of triangles $\triangle ABC$, $\triangle A'B'C'$ are equal:*

$$\angle A \cong \angle A', \quad \angle B \cong \angle B', \quad (\angle C \cong \angle C')$$

then the triangles are similar. (You need to compare only two pairs of angles, and then the third pair will be also equal)

Theorem 35 (SSS similarity test)**.** *If the corresponding sides of triangles $\triangle ABC$, $\triangle A'B'C'$ are proportional:*

$$\frac{AB}{A'B'} = \frac{AC}{A'C'} = \frac{BC}{B'C'}$$

then the triangles are similar.

Theorem 36 (SAS similarity test)**.** *If two pairs of corresponding sides of triangles $\triangle ABC$, $\triangle A'B'C'$ are proportional:*

$$\frac{AB}{A'B'} = \frac{AC}{A'C'}$$

and $\angle A \cong \angle A'$ then the triangles are similar.

Theorem 37 (RHS similarity test)**.** *If $\triangle ABC$ and $\triangle A'B'C'$ are both right-angled at B and B' respectively, and have their hypotenuse and a side proportional:*

$$\frac{AB}{A'B'} = \frac{AC}{A'C'}$$

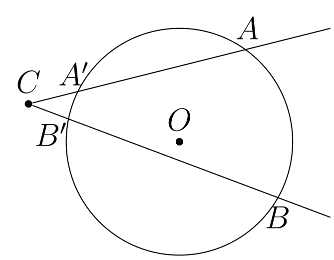
then the triangles are similar.

Proofs of all of these tests can be obtained from Thales theorem.

Homework

This homework may be more challenging than usual. Try to solve as many problems as you can, and we will discuss them all in class.

1. (A modification of Inscribed Angle Theorem.) Consider a circle λ and an angle whose vertex C is outside this circle and both sides intersect this circle at two points as shown in the figure. In this case, intersection of the angle with the circle defines two arcs: $\overset{\frown}{AB}$ and $\overset{\frown}{A'B'}$.
Prove that in this case, $m\angle C = \frac{1}{2}(\overset{\frown}{AB} - \overset{\frown}{A'B'})$.
[Hint: draw line AB' and find first the angle $\angle AB'B$. Then notice that this angle is an exterior angle of $\triangle ACB'$.]



2. Can you suggest and prove an analog of the previous problem, but when the point C is inside the circle (you will need to replace an angle by two intersecting lines, forming a pair of vertical angles)?

3. Prove Theorem 33 (using Thales Theorem). Hint: let $k = \frac{OB_1}{OA_1}$; show that then $B_iB_{i+1} = kA_iA_{i+1}$.

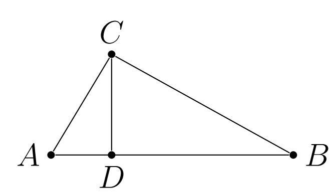
4. Using Theorem 33, describe how one can divide a given segment into 5 equal parts using ruler and compass.

5. Given segments of length a, b, c, construct a segment of length $\frac{ab}{c}$ using ruler and compass.

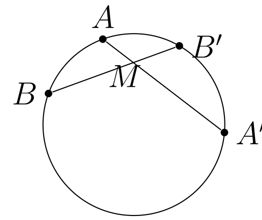
6. Let ABC be a right triangle, $\angle C = 90°$, and let CD be the altitude.

(a) Prove that $\triangle ACD \sim \triangle CBD$. Deduce from this that $CD^2 = AD \cdot DB$.

(b) Prove that both these small triangles are similar to the original triangle $\triangle ABC$. Deduce that $AC^2 = AB \cdot AD$ and $BC^2 = AB \cdot BD$. Add these to obtain a famous theorem.



7. Let M be a point inside a circle and let AA', BB' be two chords through M. Show that then $AM \cdot MA' = BM \cdot MB'$. [Hint: use inscribed angle theorem to show that triangles $\triangle AMB$, $\triangle B'MA'$ are similar.]

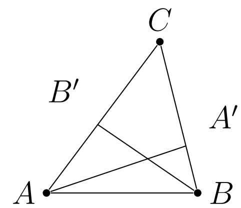


8. Let AA', BB' be altitudes in the acute triangle $\triangle ABC$.

(a) Show that points A', B' are on a circle with diameter AB.

(b) Show that $\angle AA'B' = \angle ABB'$, $\angle A'B'B = \angle A'AB$

(c) Show that triangle $\triangle ABC$ is similar to triangle $\triangle A'B'C$.



9. (Chords intersecting outside the circle). Consider circle λ, its chord AA', a point C on line (AA') outside the circle, and the tangent CD to the circle. Using similar triangles, prove that

- $|CA| \cdot |CA'| = |CD|^2$.
- for any chords AA', BB' intersecting at point C *outside* the circle, $|CA| \cdot |CA'| = |CB| \cdot |CB'|$.

Hint: connect point A to D and consider inscribed and tangent-chord angles.