MATH 8B: HANDOUT 17 [FEB 9, 2024]
EUCLIDEAN GEOMETRY 7: QUADRILATERALS WITH INSCRIBED AND CIRCUMSCRIBED CIRCLES

QUADRILATERALS AND CIRCUMSCRIBED CIRCLES

In the previous class we discussed the remarkable relation between the inscribed and central angles in any circle: the angle subtended at the center by a chord is always twice the angle subtended at any point of the circle.

From this, we easily proved the following

Theorem 27. *If a quadrilateral $ABCD$ can be circumscribed by a circle, then the sum of two opposite angles is equal to $180°$: $\angle ABC + \angle ADC = 180° = \angle BAD + \angle BCD$.*

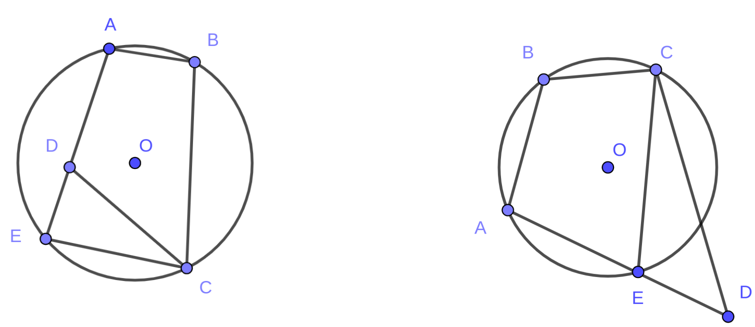
The opposite is also true, i.e. the condition for the angles above is also *sufficient* that the quadrilateral $ABCD$ can be circumscribed by a circle. This can be proven using the fact that an arc of a circle is a special *locus of points*:

Theorem 28. *For any segment AB, the locus of points C such that $\angle ACB = \alpha$ is the arc of a circle $\omega(O; |OA| = |OB|)$, for which AB is a chord, such that $\angle AOB = 2\alpha$.*

Note that there are two arcs spanned by the same circle, and the inscribed angles with vertices on these arcs are supplementary (add up to $180°$).

Proof. Indeed, any angle with vertex D inside the circle will be larger than α, and any angle with vertex outside the circle will be smaller than α. In the first case, this can be shown by continuing one side of the angle to intersection with the circle (C) and considering $\triangle ACD$; the second case is analogous. □

The application of this same argument to quadrilaterals inscribed into a circle is shown in the diagrams below.



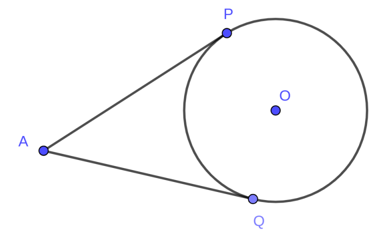
We say that a quadrilateral inscribed in a circle is a *cyclic quadrilateral*.

QUADRILATERALS WITH INSCRIBED CIRCLES

A circle inscribed into a triangle, quadrilateral, or other polygon is tangential to all the sides. The radius drawn from the center to the point common with a particular side is perpendicular to it, and is thus equal to the distance from the center to the side. The center is therefore equidistant from all the sides.

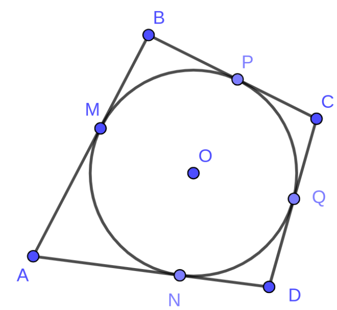
We already know that one can always inscribe a circle into a triangle: the center of the inscribed circle is the *common intersection point* of the three angle bisectors. However, a circle can be inscribed only into special quadrilaterals. Similar to the triangle, the incenter must lie on the angle bisector intersection, but it is not easy to check that all four of them intersect at the same point ("concur"). For this, we have a theorem about tangential segments to a circle drawn from from the same outside point:

Theorem 29. *Let A be a point outside a circle $\omega(O, R)$ and AP, AQ are tangential to it ($P, Q \in \omega$). Then $|AP| = |AQ|$.*

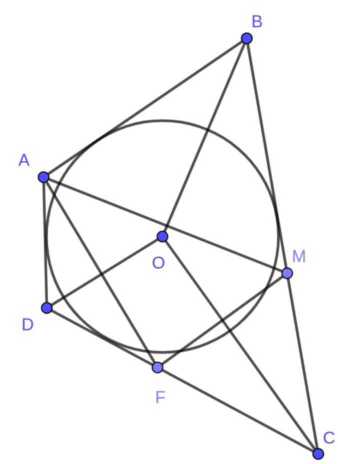


The proof is straightforward and left as an exercise. (You will need to use the RHS congruence rule.) A corollary of the proof is that O lies on the angular bisector of $\angle PAQ$.

Theorem 30. *A circle can be inscribed into a quadrilateral* $ABCD$ if and only if *the sum of lengths of two opposing sides is equal to the sum of lengths of the other two opposing sides, i.e.* $|AB| + |CD| = |AD| + |BC|$.



Proof. (1) First we prove that the condition is necessary: assume there is a circle inscribed into the quadrilateral and it has common points M, P, Q, N with sides AB, BC, CD, and AD, respectively. Then, using Theorem 29, we can immediately see that $|AM| = |AN|$, $|BM| = |BP|$, $|CP| = |CQ|$, and $|DQ| = |DN|$ and the condition is satisfied.
(2) Now let's prove that if $|AB| + |CD| = |AD| + |BC|$. First consider the case when $|AB| < |BC|$, then $|CD| > |AD|$. On side BC mark point M such that $|BM| = |BA|$ and, similarly, on side CD mark point F such that $|DF| = |DA|$. Note that also $|CF| = |CM$, and therefore $\triangle ADF$, $\triangle FCM$, and $\triangle MBA$ are isosceles.
Consider perpendicular bisectors to the sides of $\triangle AFM$: they must intersect at the same point O. At the same time, they are bisectors of angles $\angle ADC$, $\angle DCB$, and $\angle CBA$, therefore the intersection point O is *equidistant* from the four sides of quadrilateral $ABCD$ and a circle with center O can be inscribed into it. (Also, O must lie on the angle bisector of $\angle DAB$.) The proof of the case $|AB| > |BC|$ is completely analogous, and the case $|AB| = |BC|$ is left to you as homework. $\square$



The property that the sides of a triangle or a quadrilateral are eqiuidistant from the center allows one to calculate the area:

Theorem 31. *The area of a polygon with inscribed circle of radius r and perimeter P is $S = \frac{1}{2}Pr$.*

The proof is left to you as a homework.

HOMEWORK

1. Show that
- a parallelogram can be circumscribed with a circle *if and only if* it is a rectangle;
- a trapezoid $ABCD$ $AD \parallel BC$ can be circumscribed *if and only if* it is isosceles (its sides are equal $|AB| = |CD|$, or equivalently the angles at the same base are equal: $\angle DAB = \angle ADC$ or $\angle ABC = \angle DCB$).

2. Finish the proof of Theorem 30 for the case $|AB| = |BC|$.

3. Prove Theorem 31.

4. In what kind of parallelogram can a circle be inscribed?

5. Can you inscribe a circle into a quadrilateral with sides (in order)

a) 2cm, 2cm, 3cm, 3cm?

b) 5cm, 3cm, 1cm, 3cm?

c) 2cm, 5cm, 3cm, 4cm?

6. What is the area of a trapezoid with sides 5cm and 7cm and the radius of inscribed circle 2cm?

7. What is the area of a quadrilateral with two adjacent sides a and b, angle between them α, and the radius of inscribed circle r?

8. Can you circumscribe a circle around a quadrilateral $ABCD$ if the ratios of angles $\angle A : \angle B : \angle C : \angle D$ are

a) $2 : 3 : 4 : 3$?

b) $7 : 2 : 4 : 5$?

***9.** Let ABC be a triangle, and let E and F be the midpoints of AB and AC respectively. Let G and H be the foot of the altitudes from B to AC and C to AB respectively. Let $\alpha = m\angle BAC$.

(a) Show that $\triangle AEG$ is isosceles and compute the value of $m\angle HEG$. (Hint: Problem 6 of Geometry 5 handout)

(b) Similarly show that $\triangle AFH$ is isosceles and compute the value of $m\angle HFG$.

(c) Show that $HEFG$ is a cyclic quadrilateral.

(d) Extend the argument above to show that the bases of the three altitudes and the midpoints of the sides of a triangle all lie on a circle. This is called the *nine-point circle* of the triangle. (We have identified six special points on it; for the remaining three, see a geometry text or Wikipedia.)

(e) Let O be the circumcenter of $\triangle ABC$ (the point of intersection of perpendicular bisectors of the sides), and let Q be the orthocenter (the point of intersection of the altitudes). Consider the quadrilateral $GFOQ$. Show that it is a trapezoid. Hence deduce that the center of the nine-point circle is the midpoint of OQ.