MATH 8B: HANDOUT 16 [FEB 2, 2024]
EUCLIDEAN GEOMETRY 6: CIRCLES

CIRCLES

Definition. A circle with center O and radius $r > 0$ is the set of all points P in the plane such that $OP = r$.

Traditionally, one denotes circles by Greek letters: $\lambda, \omega \dots$.
Given a circle λ with center O,

- A radius is any line segment from O to a point A on λ,
- A chord is any line segment between distinct points A, B on λ,
- A diameter is a chord that passes through O,
- A line is tangent if it intersects the circle at one point, and is said to be the tangent through that point.
- Two circles are tangent if they intersect at exactly one point.

Theorem 21. *Let A be a point on circle λ centered at O, and m a line through A. Then m is tangent to λ if and only if $m \perp \overline{OA}$. Moreover, there is exactly one tangent to λ at A.*

Proof. First we prove (m is tangent to λ) $\implies$ ($m \perp \overline{OA}$). Suppose m is tangent to λ at A but not perpendicular to $\overline{OA}$. Let $\overline{OB}$ be the perpendicular to m through O, with B on m. Construct point C on m such that $BA = BC$; then we have that $\triangle OBA \cong \triangle OBC$ by SAS, using $OB = OB$, $\angle OBA = \angle OBC = 90°$, and $BA = BC$. Therefore $OC = OA$ and hence C is on λ. But this means that m intersects λ at two points, which is a contradiction. Now we prove ($m \perp \overline{OA}$) $\implies$ (m is tangent to λ). Suppose m passes through A on λ such that $m \perp \overline{OA}$. If m also passed through B on λ, then $\triangle AOB$ would be an isosceles triangle since $\overline{AO}$, $\overline{BO}$ are radii of λ. Therefore $\angle ABO = \angle BAO = 90°$, i.e. $\triangle AOB$ is a triangle with two right angles, which is a contradiction. □

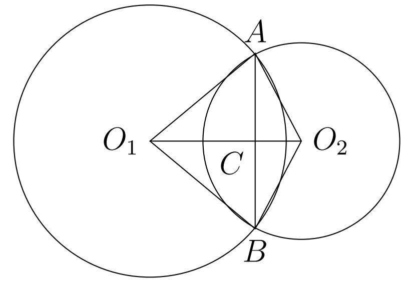
Notice that, given point O and line m, the perpendicular $\overline{OA}$ from O to m (with A on m) is the shortest distance from O to m, therefore the locus of points of distance exactly OA from O should line entirely on one side of m. This is essentially the idea of the above proof.

Theorem 22. *Let $\overline{AB}$ be a chord of circle λ with center O. Then O lies on the perpendicular bisector of $\overline{AB}$. Moreover, if C is on $\overline{AB}$, then C bisects $\overline{AB}$ if and only if $\overline{OC} \perp \overline{AB}$.*

Proof. Let m be the perpendicular bisector of $\overline{AB}$. The center O of λ is equidistant from A, B by the definition of a circle, therefore O must be on m. Let m intersect $\overline{AB}$ at D. We then have that D is the midpoint of $\overline{AB}$ and also the foot of the perpendicular from O to $\overline{AB}$ (that is, $\overline{OD} \perp \overline{AB}$).

Then if C bisects $\overline{AB}$, $C = D$ since D is the midpoint of $\overline{AB}$, and it follows that $\overline{OC} = \overline{OD} \perp \overline{AB}$. Conversely, if C is on $\overline{AB}$ with $\overline{OC} \perp \overline{AB}$, then because there is only one perpendicular to $\overline{AB}$ through O, we must have that the lines $\overline{OC}$ and $\overline{OD}$ coincide, and therefore their intersection points with $\overline{AB}$ must be the same: $C = D$. Therefore C is the midpoint of $\overline{AB}$. □

Theorem 23. *Let ω_1, ω_2 be circles with centers at points O_1, O_2 that intersect at points A, B. Then $\overline{AB} \perp \overline{O_1O_2}$.*



Proof. Let l be the perpendicular bisector of AB. By the previous theorem, l contains both centers: $O_1 \in l$, $O_2 \in l$. Thus, $l = \overline{O_1O_2}$, so $\overline{O_1O_2}$ is the perpendicaulr bisector of AB; in particular, they are perpendicular. □

Theorem 24. *(Relative positions of lines and circles) Let λ be a circle of radius r with center at O and let l be a line. Let d be the distance from O to l, i.e. the length of the perpendicular OP from O to l. Then:*

- *If $d > r$, then λ and l do not intersect.*
- *If $d = r$, then λ intersects l at exactly one point P, the base of the perpendicular from O to l. In this case, we say that l is tangent to λ at P.*
- *If $d < r$, then λ intersects l at two distinct points.*

Proof. The first two parts easily follow from the fact that a perpendicular is the (shortest) distance from a point to a line. In the last part, it is easy to show that λ can not intersect l at more than 2 points. It is also easy to show that if λ and l intersect in at least one point, then they have two points of intersection. Proving that there is a point of intersection is rather subtle: it requires some notion of continuity of the real numbers and is tantamount to an additional postulate (for example, saying that if l contains a point inside the circle λ, then they must have a point of intersection). We will not go into this discussion here. □

Note that it follows from the definition that a tangent line is perpendicular to the radius OP at point of tangency. Converse is also true.

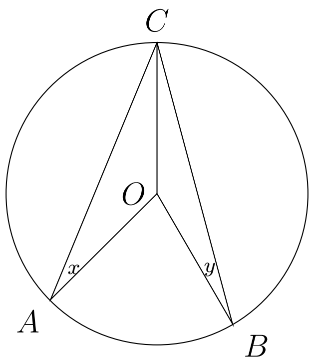
Theorem 25. *Let ω_1, ω_2 be circles that are both tangent to line m at point A. Then these two circles have only one common point, A. Such circles are called* tangent.

Proof. By Theorem 21, radiuses O_1A and O_2A are both perpendicular to m at A; since there can only be one perpendicular line to m at given point, it means that O_1, O_2, and A are on the same line, and that m is perpendicular to O_1O_2 at A.

Now, suppose that ω_1, ω_2 intersect at point $B \neq A$. Then by the previous theorem, $\overline{AB} \perp \overline{O_1O_2}$, therefore both $\overline{AB}$ and m are perpendicular to $\overline{O_1O_2}$ through A. We must therefore have that B is on m, but m is tangent to ω_1 through A, thus has only one intersection with ω_1, which is a contradiction. □

Arcs and Angles

Consider a circle λ with center O, and an angle formed by two rays from O. Then these two rays intersect the circle at points A, B, and the portion of the circle inside this angle is called the arc subtended by $\angle AOB$.



Theorem 26. *Let A, B, C be on circle λ with center O. Then $\angle ACB = \frac{1}{2}\angle AOB$. The angle $\angle ACB$ is said to be inscribed in λ.*

Proof. There are actually a few cases to consider here, since C may be positioned such that O is inside, outside, or on the angle $\angle ACB$. We will prove the first case here, which is pictured on the left.
Case 1. Draw in segment $\overline{OC}$. Denote $m\angle A = x$, $m\angle B = y$. Since $\triangle AOC$ is isosceles, $m\angle AC) = x$; similarly $m\angle BCO = y$, so $m\angle ACB = x + y$, and $m\angle AOC = 180° - 2x$, $m\angle BOC = 180° - 2y$. Therefore, $m\angle AOC + m\angle BOC = 360° - 2(x + y)$. This implies $m\angle AOB = 2(x + y)$. □

As a result of Theorem 26, we get that any triangle $\triangle ABC$ on λ where $\overline{AB}$ is a diameter must be a right triangle, since the angle $\angle ACB$ has half the measure of angle $\angle AOB$, which is $180°$.
The idea captured by the concept of an arc and Theorem 26 is that there is a fundamental relationship between angles and arcs of circles, and that the angle $360°$ can be thought of as a full circle around a point.

HOMEWORK

1. Prove that, given a segment $\overline{AB}$, there is a unique circle with diameter $\overline{AB}$.
2. Given lines $\overleftrightarrow{AB} \| \overleftrightarrow{CD}$ such that $\overline{AD}$, $\overline{BC}$ intersect at E and $AE = ED$, prove that $BE = EC$.
3. Prove that if a diameter of circle λ is a radius of circle ω, then λ, ω are tangent.
4. Complete the proof of Theorem 26 by proving the cases where O is not inside the angle $\angle ACB$. [Hint: for one of the cases, you may need to write $\angle ACB$ as the difference of two angles.]
5. Prove the converse of Theorem 26: namely, if λ is a circle centered at O and A, B, are on λ, and there is a point C such that $m\angle ACB = \frac{1}{2}m\angle AOB$, then C lies on λ. [Hint: we need to prove that $OC = OA$; consider using a proof by contradiction]
6. Let A, B be on circle λ centered at O and m the tangent to λ at A. Let C be on m such that C is on the same side of $\overleftrightarrow{OA}$ as B. Prove that $m\angle BAC = \frac{1}{2}m\angle BOC$. [Hint: extend $\overline{OA}$ to intersect λ at point D so that $\overline{AD}$ is a diameter of λ. What arc does $\angle DAB$ subtend?]
7. Prove that, given two distinct points A, B on circle λ which are on the same side of diameter $\overline{CD}$ of λ, that $CB \neq CA$.
8. Let $\overline{AB}$, $\overline{CD}$ both have midpoint E and let F, G be points such that $BECF$ and $AEDG$ are parallelograms. Prove that E is the midpoint of FG.