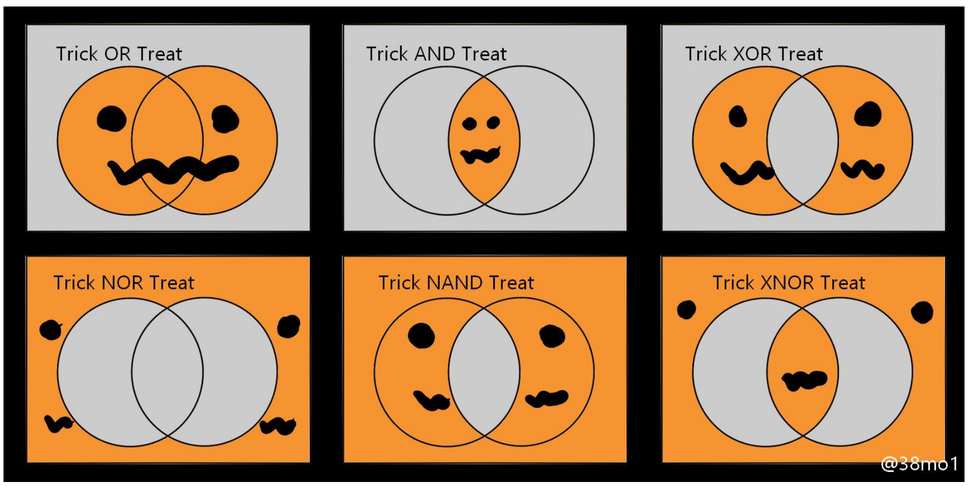
MATH 8B [2024 OCT 27]
HANDOUT 6 : LOGIC 1 : INTRODUCTION TO SYMBOLS AND FORMULAS

Today we will start discussing formal rules of logic. In logic, we will be dealing with *boolean* expressions, i.e. expressions which only take two values, `TRUE` and `FALSE`. We will commonly use abbreviations T and F for these values.

You can also think of these two values as the two possible digits in binary (base 2) arithmetic: $T = 1$, $F = 0$.

In the usual arithmetic, we have some operations (addition, multiplication, ...) which satisfy certain laws (associativity, distributivity, ...). Similarly, there are logic operations and logic laws.

The branch of mathematics which deals with the laws and relationships between logical variables, using logical operators, is called Boolean algebra.[1]



Operations in Logic

Basic operations.

- NOT (for example, NOT A): true if A is false, and false if A is true. Commonly denoted by $\neg A$ or (in computer code) $!A$. I also like to use $\bar{A}$.
- AND (for example A AND B): true if both A, B are true, and false otherwise (i.e., if at least one of them is false). Commonly denoted by $A \wedge B$. Another shorthand is $A \cdot B$ or just AB.
- OR (for example A OR B): true if at least one of A, B is true, and false otherwise. Sometimes also called "inclusive or" to distinguish it from the "exclusive or" described in problem 4 below. Commonly denoted by $A \vee B$. Another shorthand is $A + B$.

As in usual algebra, logic operations can be combined, e.g. $(A \vee B) \wedge C$. (which is $(A + B) \cdot C$ in shorthand)

[1] After George Boole, an English mathematician, logician, and philosopher, author of *The Laws of Thought*.

Some other logical operations.

- NAND(A,B) = NOT(AND(A,B))
- NOR(A,B) = NOT(OR(A,B))
- XOR(A,B) (see problem 4)
- XNOR(A,B) = NOT(XOR(A,B))

TRUTH TABLES

If we have a logical formula involving variables $A, B, C, \dots$, we can make a table listing, for every possible combination of values of $A, B, \dots$, the value of our formula. For example, the following is the truth tables for OR and AND:

A	B	A OR B
T	T	T
T	F	T
F	T	T
F	F	F

A	B	A AND B
T	T	T
T	F	F
F	T	F
F	F	F

A truth table for an expression involving n variables is going to have 2^n rows.

LAWS OF LOGIC

We can combine logic operations, creating more complicated expressions such as $A \wedge (B \vee C)$. As in arithmetic, these operations satisfy some laws: for example $A \vee B$ is the same as $B \vee A$. Here, "the same" means "for all values of A, B, these two expressions give the same answer"; it is usually denoted by $\iff$. (Sometimes, I will abbreviate the $\iff$ to $=$.)

Truth tables provide the most straightforward (but not the shortest) way to prove complicated logical rules: if we want to prove that two formulas are equivalent (i.e., always give the same answer), make a truth table for each of them, and if the tables coincide, they are equivalent.

Here is a list of some useful laws of Boolean algebra. They can all be proved using truth tables, but sometimes there is a quicker way to see them. In the problems, you will verify some of these using truth tables.

- Commutative Law:

$$A \vee B \iff B \vee A$$
$$A \wedge B \iff B \wedge A$$

- Associative Law:

$$A \wedge (B \wedge C) \iff (A \wedge B) \wedge C$$
$$A \vee (B \vee C) \iff (A \vee B) \vee C$$

- Distributive Law:

$$A \wedge (B \vee C) \iff (A \wedge B) \vee (A \wedge C)$$
$$A \vee (B \wedge C) \iff (A \vee B) \wedge (A \vee C)$$

- De Morgan's Laws:

$$\neg(A \vee B) \iff \neg A \wedge \neg B$$
$$\neg(A \wedge B) \iff \neg A \vee \neg B$$

- Idempotence:

$$A \vee A \iff A$$
$$A \wedge A \iff A$$

- Identity:

$$A \wedge T \iff A$$
$$A \vee F \iff A$$

- Domination:

$$A \wedge F \iff F$$
$$A \vee T \iff T$$

- Excluded middle:

$$A \vee \neg A \iff T$$
$$A \wedge \neg A \iff F$$

Duality

In Boolean algebra, frequently there is an obvious duality between $\vee$ and $\wedge$; for example, if you have an identity with expressions involving these operators, you can switch them (while also switching F and T). For example, applying this switching operator takes you from the identity

$$A \wedge (B \vee C) \iff (A \wedge B) \vee (A \wedge C)$$

(the first part of the distributive law) to

$$A \vee (B \wedge C) \iff (A \vee B) \wedge (A \vee C)$$

[To prove this duality; replace all variables/truth values in the expressions by their negations, nnegate both sides of the identity in question, and use de Morgan's Laws.]

Logic puzzles with Boolean algebra

Suppose you are on the island of knights and knaves; knights always tell the truth and knaves always lie. You meet two people, A and B. A says: we (A and B) are both knaves. Can you determine if A is a knight or knave, and same for B?

You can obviously reason through first principles (if A is a knight then his statement is true, therefore ...). This is equivalent in some way to exploring an appropriate truth table. But we can also use a bit of Boolean algebra, for example as follows:

Let x be the boolean variable that A is a knight. Let y be the Boolean variable that B is a knight. Let z be A's statement, namely

$$z = \bar{x} \cdot \bar{y}$$

We know that

$$x \iff z,$$

so

$$x = \bar{x}\bar{y}$$

multiply x into both sides (i.e. take $\wedge$ with x), and use the law of associativity, idempotence, excluded middle, and domination, to get

$$\begin{aligned} x \cdot x &= x \cdot \bar{x} \cdot \bar{y} \\ x &= F \cdot \bar{y} \\ x &= F \end{aligned}$$

So $\bar{x} = T$, and the first equivalence yields

$$F = x = T \cdot \bar{y} = \bar{y}$$

Therefore $y = T$. So A is a knave and B is a knight.

IF-THEN STATEMENTS

If-then statements or implications are expressions of the form "if A is true then B is true". They are written as $A \Rightarrow B$. Let's see, with the aid of a truth table, that it is equivalent to $\neg A \vee B$.

A	B	$A \Rightarrow B$	$\neg A \vee B$
T	T	T	T
T	F	F	F
F	T	T	T
F	F	T	T

The only possibly confusing line here is the last one; remember, if A is false, then the statement "if A is true then B is true" is vacuously true (whether or not B turns out to be true).

In terms of a Venn diagram, $A \Rightarrow B$ represents the union of the region corresponding to B and the complement of the region corresponding to A. So it is True if this union is the whole universe, i.e. if A's region is contained in B's. From this standpoint, it is intuitively clear that $A \Rightarrow B$ and $B \Rightarrow C$ imply $A \Rightarrow C$. Problem 10 asks you to check this with Boolean algebra.

PROBLEMS

1. Write the truth table for each of the following formulas. Are they equivalent (i.e., do they always give the same value)?
   (a) $(A \vee B) \wedge (A \vee C)$
   (b) $A \vee (B \wedge C)$.

2. Use the truth tables to prove *De Morgan's laws*

$$\neg(A \wedge B) \iff (\neg A) \vee (\neg B)$$
$$\neg(A \vee B) \iff (\neg A) \wedge (\neg B)$$

3. Use truth tables to show that $\vee$ is commutative and associative:

$$A \vee B \iff B \vee A$$
$$A \vee (B \vee C) \iff (A \vee B) \vee C$$

Is it true that $\wedge$ is also commutative and associative?

4. Another logic operation, called "exclusive or", or XOR, is defined as follows: A XOR B is true if and only if exactly one of A, B is true.

(a) Write a truth table for XOR

(b) Describe XOR using only basic logic operations AND, OR, NOT, i.e. write a formula using variable A, B and these basic operations which is equivalent to A XOR B.

5. Yet one more logic operation, NAND, is defined by

$$A \text{ NAND } B \iff \text{NOT}(A \text{ AND } B)$$

(a) Write a truth table for NAND

(b) What is A NAND A?

*(c) Show that you can write NOT A, A AND B, A OR B using only NAND (possibly using each of A, B more than once).

This last part explains why NAND chips are popular in electronics: using them, you can build **any** logical gates.

6. A restaurant menu says *The fixed price dinner includes entree, dessert, and soup or salad*.

Can you write it as a logical statement, using the following basic pieces:
E: your dinner includes an entree
D: your dinner includes a dessert
P: your dinner includes a soup
S: your dinner includes a salad
and basic logic operations described above?

7. On the island of knights and knaves, there are two kinds of people: Knights, who always tell the truth, and Knaves, who always lie. Unfortunately, there is no easy way of knowing whether a person you meet is a knight or a knave...

You meet two people on this island, Bart and Ted. Bart claims, "I and Ted are both knights or both knaves." Ted tells you, "Bart would tell you that I am a knave." So who is a knight and who is a knave?

***8.** (a) How many distinct logical functions are there of n different logical variables? [Hint: How many truth tables are there?]

(b) Can all of them be expressed with the logical operators $\wedge, \vee, \neg$ and the literals T and F? [Hint: check this first for one variable A. Then check this for two variables A and B by splitting the truth table in half, corresponding to A being T or F. Then try to do the same for more variables.]

(c) Can all of them be expressed by just the logical operator NAND?

9. Show that $A \vee (\neg A \wedge B) \iff A \vee B$.

***10.** Show that $A \Rightarrow B$ and $B \Rightarrow C$ imply $A \Rightarrow C$.